American University in Bulgaria

Department of Economics

A Senior Thesis in Economics

Granularity Adjustments (GA) For Regulatory Capital: Testing Reparameterized GA of Extended Single-Factor CreditRisk+

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Abstract

The regulatory model for capital requirements under the first pillar of the Basel Accords assumes that portfolios are perfectly fine-grained and there exists no concentration risk from large exposures. Granularity Adjustment (GA) methodologies provide a way of assessing the amount of additional capital, which would be needed to account for undiversified concentration risk. Gordy and Lütkebohmert (2013) propose a closed-form expression that is derived within an Extended Single-Factor CreditRisk+ model and is calibrated to representative data from 38 German banks. We use a simulation framework to examine the behavior of their GA formulation against an actuarial CreditRisk+ model, in which obligors carry default risk only. Our findings show that the Gordy and Lütkebohmert (2013) specification approximates relatively well GA for large, diversified portfolios. In smaller and more granular portfolios that also carry the largest amount of non-accounted for concentration risk, however, the formula leads to a significant underestimation of GA, which calls for special regulatory attention under the second pillar of the Basel framework.

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1 Introduction

One of the main goals of banking regulation is to determine the optimal amount of regulatory capital for banks. On the one hand, banks must hold enough capital to be able to withstand an unlikely event of large losses in their portfolios; on the other hand, capital requirements should not be so large as to force inefficiencies upon banks by freezing up unnecessarily large amounts of capital. Thus, under Basell II and III accords capital requirements are determined in a way that ensures banks are able to withstand all losses except the most extreme outcomes with cumulative probability $1-q$, where q is set by regulators. In technical terms, for a given portfolio of size n, banks are required to have enough financial resources to be able to endure any loss amount (a realization of a random variable L_n) up to the qth quantile of the probability distribution of portfolio loss L_n (which is called q-Value-at-Risk of L_n (or $VaR_q(L_n)$) in credit risk terminology). To meet this end, capital requirements are determined by $VaR_q(L_n) - E[L_n]$, where $E[L_n]$ serves as the basis for reserve requirements.

An exact expression for $VaR_q(L_n)$ is model specific. To ensure portfolio-invariant[1] capital rules, Basel II and III rely on Internal Ratings Based (IRB) Asymptotic Single Risk Factor (ASRF) model (the Basel IRB[2] model henceforth) for determining $VaR_q(L_n)$ in a regulatory framework. ASRF models assume that a vector of systemic factors is one-dimensional (i.e. a single global business cycle is the source of all systemic risk) and that portfolios are infinitely granular, i.e. they consist of infinitely many borrowers with arbitrarily small exposure shares in the limit. While the single-factor assumption does not usually result in significant distortions of capital estimation, the assumption of infinite granularity can be problematic. Tarashev and Zhu (2008), for instance, find that while single risk factor assumption leads to underprediction of target capital by 1% only, infinite granularity assumption can lead to underestimation of capital by 19%. This is because infinite granularity implies a perfectly diversified portfolio, in which idiosyncratic risk (risk specific to individual borrowers as opposed to systemic risk that affects all borrowers) is completely eliminated, and whose loss amount is uniquely determined by a systemic variable X. In short, ASRF models assume that L_n and

[1] i.e. a rule should be such that the amount of capital to be held for any exposure depends on that exposure only and not on the characteristics of the portfolio in which it is held. This is because portfolio-variant rules are too complicated to implement for most financial institutions as well as regulators.

[2] Under the standard IRB approach of Basel II and III, banks are allowed to use their own estimations of probability of default (PD) and loss given default (LGD) for computing minimum capital requirements using Basel formula.

$E[L_n|X]$ are identically distributed, and that $VaR_q(L_n) = VaR_q(E[L_n|X])$. For large, well diversified portfolios $VaR_q(L_n) - VaR_q(E[L_n|X])$ is expected to be negligible, but for portfolios in which a single borrower (or a few borrowers) constitute a significant part of total exposure, the error can be substantial. For this reason, there is a need for Granularity Adjustment (GA) formulas[3] to account for undiversified idiosyncratic risk in a regulatory setting.

First ever granularity adjustment formula appeared in the Second Consultative Paper (CP2) by Basel Committee on Banking Supervision (2001) and was essentially an empirical result: it was obtained by fitting a functional form to the difference between "true" (simulated) VaR and its asymptotic approximation for a number of simulated portfolios. Building on the results of Gourieroux, Laurent, and Scaillet (2000), Martin and Wilde (2002) provided a first ever theoretical derivation of GA, which was approximated using a second-order Taylor-series expansion and is similar in form to CP2 GA. Voropaev (2011) derived GA using Hermit polynomials, and obtained one additional term in the GA expansion.

GAs mentioned above are approximations derived in a general setting and their specific algebraic form depends on a credit risk model. In a regulatory setting, it would be natural to base GA on the Basel IRB model that underpins Basel capital formula. However, the IRB model is of mark-to-market type and Gordy and Marrone (2012) demonstrate that for such models GA has to be estimated using simulations and cannot be obtained analytically. Gordy and Lütkebohmert (2013), on the other hand, derive GA for an actuarial model of Extended Single-factor CreditRisk+, which has a relatively simple closed-form expression[4] that can be reparametrized so that it is expressed only in terms of data inputs that are already required from banks for reporting minimum capital requirements. Further, they calibrate the Creditrisk+ GA formula so that it matches the GA[5] implied by the IRB model on a large representative portfolio built from the data sample of 38 German bank portfolios. The authors further

[3]Note that GAs can also be estimated using simulations. Closed form expressions, however, are much more fit for regulatory application, since simulations pose significant implementation and verification challenges.

[4]Gordy and Lütkebohmert (2013) do not make use of Voropaev's additional term in the expansion. This is perhaps due to the fact that the third-order correction term contains the third central moment of the distribution of $(L_n - E[L_n|X])$, whose computation poses additional challenges and might not be obtainable in closed-form within CreditRisk+.

[5]GA implied by the IRB model is further estimated using Creditmetrics, which is very close in spirit to the Basel IRB Model. The estimation of GA using Creditmetrics relies on simulations and is performed using the approach of Gordy and Marrone (2012).

empirically evaluate their formula for these 38 portfolios and find that it is well-behaved with respect to portfolio size and diversification - on average it provides small, moderate, and substantial adjustments for large, medium, and small portfolios, respectively. For its simplicity, ease of implementation and supposed precision, Gordy and Lütkebohmert propose regulatory use of the Creditrisk+ GA.

Its nice behavior notwithstanding, the CreditRisk+ GA can have two issues. Firstly, the authors have shown that their GA is well-behaved on the very portfolios based on which it is calibrated. Thus, it is not necessary that it will do well on portfolios of different characteristics. Secondly, under mark-to-market models, losses occur in case of not only defaults but also rating grade transitions. However, many portfolios (e.g. retail and SME ones) consist of loans that are untradeable and/or do not carry rating grades. Since these portfolios carry only default risk, it is unclear whether mark-to-market model-based calibration of the GA will perform well. The focus of this paper is the latter issue.

Our goal is to compare the GA of Gordy and Lütkebohmert (2013) against the GA implied by an actuarial credit risk model. Our methodology is described in Section 1.1. Section 2 discusses the mathematical framework of credit risk modeling, explains the probem of granularity in greater detail and exposes the features of CreditRisk+ as well as of the GA of Gordy and Lütkebohmert. Section 3 describes the construction of portfolios, calibration of CreditRisk+ for simulations and our algorithm for Monte Carlo Simulations. Section 4 discusses the results and Section 5 concludes.

1.1 Methodology

To test the GA of Gordy and Lütkebohmert ($\tilde{GA}$ henceforth) we simulate 8 portfolios of varying sizes and granularity based on aggregate data on 38 German bank portfolios found in Gordy and Lütkebohmert (2013). Details are provided in Sections 3.1 and 3.2. Our goal is to evaluate $\tilde{GA}$ for these portfolios and compare it against GA implied by an actuarial credit risk model. For the choice of the model, Restricted (Two-State) CreditMetrics is a natural candidate, since its mark-to-market generalization is very close to the IRB model. That said, Gordy (2000) shows using simulations that Restricted CreditMetrics and CreditRisk+ produce very similar loss distributions for appropriate calibrations of CreditRisk+[6]. For its relative simplicity, we thus do simulations on

[6]Note that this applies to Restricted (i.e. Actuarial) CreditMetrics only, and not to its mark-to-market generalization.

Extended Single-Factor CreditRisk+.

Since $\tilde{GA}$ has a closed-form expression (see Section 2.4), its evaluation is straightforward. For CreditRisk+, we need to obtain $VaR_q(L_n)$ and $VaR_q(E[L_n|X])$, which will give us granularity adjustment $GA = VaR_q(L_n) - VaR_q(E[L_n|X]) \equiv \alpha_q(L_n) - \alpha_q(E[L_n|X])$. In Section 2.3.1 we prove that for CreditRisk+ $\alpha_q(E[L_n|X]) = E[L_n|X = \alpha_q(X)]$ and can thus be obtained analytically. $\alpha_q(L_n)$, on the other hand, is estimated using Monte Carlo Simulations, whose algorithm and calibration is provided in Section 3.3.

2 Mathematical Framework

In this section we discuss the mathematical farmework of credit risk modeling. Section 2.1 describes the general setting and discusses the usefuleness of ASRF models and their potential shortcomings. Section 2.2 presents briefly a formal way to measure the granularity of a portfolio. Section 2.3 exposes the summary of Extended Single-factor CreditRisk+ and derives its asymptotic capital requirements. Lastly, Section 2.4 presents the summary of the derivation of the GA proposed by Gordy and Lütkebohmert (2013).

2.1 General Setting

In this section we expose the general mathematical framework of credit risk modeling and provide a technical motivation for the need of granularity adjustments.

Let A_i represent the nominal Exposure at Default (EAD) to ith borrower in a portfolio of n loans. The stochastic loss function is then given by

$$L_n = \sum_{i=1}^{n} s_i U_i,$$

where $s_i = \frac{A_i}{\sum_{i=1}^{n} A_i}$ and $U_i \in [0,1]$ is the stochastic loss rate on ith exposure. A vector of systemic factors X generates the correlation across exposure losses, where the specific correlation structure depends on a credit risk model. Conditional on X, $\{U_i\}$ is a mutually independent set. qth quantile of L_n (or $VaR_q(L_n)$) is usually denoted by $\alpha_q(L_n)$. $\mathcal{R}$ and $\mathcal{K}$ denote portfolio level reserve and capital requirements, respectively, i.e. $\mathcal{R} = E[L_n]$ and $\mathcal{K} = \alpha_q(L_n) - \mathcal{R} = \alpha_q(L_n) - E[L_n]$.

Gordy (2003) proves that if $\{A_i\}$ is a sequence of positive constants such that

$\sum_{i=1}^{n} A_i \uparrow \infty$ and $\frac{A_n}{\sum_{i=1}^{n} A_i} \to 0$[7] as $n \to \infty$, then $L_n \to E[L_n|X]$ and $\alpha_q(L_n) \to \alpha_q(E[L_n|X])$. Naturally, the preconditions of this result imply infinite granularity of a portfolio, since from them it follows that $s_i \to 0$ as $n \to \infty$. Further, Gordy also proves that if X is one-dimensional and if some additional mild regularity conditions are satisfied, then for large enough n, $\alpha_q(E[L_n|X]) = E[L_n|X = \alpha_q(X)]$[8]. This result is extremely convenient, since it allows expression of portfolio level capital requirements as the sum of instrument level *portfolio-invariant* capital rules. To see this, let $\mathcal{K}_i$ denote asymptotic[9] capital requirement for ith exposure, and observe that

$$\begin{aligned}\mathcal{K} &= \alpha_q(L_n) - E[L_n] \approx E[L_n|X = \alpha_q(X)] - E[L_n] = \\ &= \sum_{i=1}^{n} s_i E[U_i|X = \alpha_q(X)] - \sum_{i=1}^{n} s_i E[U_i] = \\ &= \sum_{i=1}^{n} s_i \left(E[U_i|X = \alpha_q(X)] - E[U_i]\right) = \sum_{i=1}^{n} s_i \mathcal{K}_i.\end{aligned}$$

Since the convergence of the conditional and unconditional loss distributions is an asymptotic result that relies on infinite granularity of a portfolio, the approximation $\alpha_q(L_n) \approx \alpha_q(E[L_n|X])$ is not necessarily good for coarse-grained portfolios of modest size. To give a heuristic illustration of this point, observe that by conditional independence of U_i,

$$\begin{aligned}Var(L_n|X) = Var(\sum_{i=1}^{n} s_i U_i|X) = \\ = \sum_{i=1}^{n} s_i^2 Var(U_i|X) \leq \\ \leq \max_i\{Var(U_i|X)\} \times \sum_{i=1}^{n} s_i^2.\end{aligned}$$

Since $U_i \in [0,1]$, $\max_i\{Var(U_i|X)\}$ is a small number even for high values of X (e.g. for a high conditional standard deviation of 0.3 corresponding to some downturn realization x of X, $Var(U_i|X = x) = 0.09$). Thus, for a small enough $\sum_{i=1}^{n} s_i^2$, $Var(L_n|X)$ is

[7] Gordy (2003) also imposes conditions on the rate of convergence, which we skip here for simplicity.

[8] For Single-factor Creditrisk+, we will give a direct proof of the statement in Section 2.3.1. Moreover, we will see that for Creditrisk+ the equality holds for any n.

[9] i.e. capital requirements as approximated using $\alpha_q(E[L_n|X])$.

negligible[10] , i.e. L_n is almost uniquely determined by X.[11] In other words, given a realization of a systemic variable, idiosyncratic factors add very little variability to the loss outcome. This, however, is not necessarily true for less diversified portfolios with higher $\sum_{i=1}^{n} s_i^2$. In fact, $\sum_{i=1}^{n} s_i^2$ is a widely used measure of risk concentration and is known as the Herfindahl-Hirschman Index (HHI). Since we will use HHI for generating a simulated portfolio, we discuss it briefly next.

2.2 Herfindahl-Hirschman Index (HHI)

In this section we are going to explain briefly why HHI is a good measure of granularity (or equivalently of diversification). By construction, the range of HHI is given by $[0, 1]$. Since any finite portfolio has a non-zero HHI, an index of 0 corresponds uniquely to the infinitely granular portfolio[12]. Further, if for any finite portfolio $\max_i\{s_i\} < 1$, $\sum_{i=1}^{n} s_i^2 < 1$. Thus, HHI of 1 corresponds to a single borrower, or the least diversified portfolio. As for the cases in between, we will demonstrate that for fixed n, less diversified portfolios have higher HHI. More precisely, we will show that for any portfolio, decreasing any of its exposure shares at the expense of increasing a larger exposure share raises HHI. This is formalized in the Proposition that follows.

Proposition 2.1. *Let $\mathscr{P}$ be an arbitrary portfolio of size n with exposure ratios $\{s_i\}_{i\leq n}$ and assume $s_1 \leq s_2$. Let $\mathscr{P}^*$ be another portfolio of size n and exposure ratios $\{s_i^*\}_{i\leq n}$, such that $s_i^* = s_i$ for $3 \leq i \leq n$, and $s_1^* = s_1 - a$ and $s_2^* = s_2 + a$, for some $0 < a < s_1$. Then, $HHI_{\mathscr{P}} < HHI_{\mathscr{P}^*}$.*

Proof. $HHI_{\mathscr{P}} - HHI_{\mathscr{P}^*} = [s_1^2 + s_2^2 + \sum_{i=3}^{n} s_i^2] - [(s_1 - a)^2 + (s_2 + a)^2 + \sum_{i=3}^{n} s_i^2] = (s_1^2 + s_2^2) - (s_1^2 - 2as_1 + a^2) - (s_2^2 + 2as_2 + a^2) = 2a(s_1 - s_2) - 2a^2 \leq -2a^2 < 0.$ □

Note that for a homogeneous portfolio $s_i = s_j$ for any $i, j \in \{1, 2, ..., n\}$. Thus, any deviation from a homogeneous case produces a portfolio with higher HHI. In other

[10]For instance, Gordy and Lütkebohmert (2013) find that $\sum_{i=1}^{n} s_i^2$ is at most 0.0012 for 4 German bank portfolios with exposure count within 4,000-8,999. Assuming $\max_i\{Var(U_i|X = x)\} = 0.09$, it follows that for these portfolios $Var(L_n|X = x) \leq 0.0001$, i.e. given $X = x$, on average L_n deviates from $E[L_n|X = x]$ by at most 0.01.

[11]Note that the rate of convergence imposed on $\frac{A_n}{\sum_{i=1}^{n} A_i}$ by Gordy (2003) guarantees that $\sum_{i=1}^{n} s_i^2 \to 0$ as n becomes large. Thus, since U_i is bounded, $\max_i\{Var(U_i|X)\} \times \sum_{i=1}^{n} s_i^2 \to 0$, i.e. $Var(L_n|X) = 0$ in the limit. This shows that idiosyncratic factors do not affect the loss outcome in the limit.

[12]The rate of convergence imposed on $\frac{A_n}{\sum_{i=1}^{n} A_i}$ by Gordy (2003) guarantees that $\sum_{i=1}^{n} s_i^2 \to 0$ as n becomes large.

words, for any n, a homogeneous portfolio has the smallest HHI. This agrees with the intuition that among finite portfolios the homogeneous one is the most diversified.

2.3 Single-Factor CreditRisk+

In Creditrisk+ $U_i = LGD_i 1_{D_i}$, i.e. a portfolio loss function is given by

$$L_n = \sum_{i=1}^{n} s_i LGD_i 1_{D_i},$$

where $s_i = \frac{A_i}{\sum_{i=1}^{n} A_i}$ and A_i is deterministic (not random). Loss Given Default (LGD_i) is a loss rate on ith exposure in case of default, and 1_{D_i} is a default indicator assuming a value of 1 if the ith borrower defaults and 0 otherwise. More precisely, 1_{D_i} is a Bernoulli distributed random variable with "success" probability $\pi_i(x)$, where

$$\pi_i(x) = PD_i(1 + w_i(x - 1).$$

x is a realization of a one-dimensional systemic factor X, which is assumed to be Gamma-distributed with mean 1 and variance $\frac{1}{\xi}$.[13] $0 < w_i \leq 1$ is a factor loading measuring the sensitivity of ith borrower to a systemic variable. $PD_i = E(\pi_i(X))$ is an unconditional default probability of ith exposure. Conditional on X, default events are independent. In order to obtain VaR in analytic form, Bernoulli distribution is approximated using Poisson distribution. For a detailed analysis of Creditrisk+, see (Gordy 1999).

The standard CreditRisk+ for which a closed-form solution is available assumes fixed LGD_i and does not therefore account for a potentially important source of volatility. For this reason, Gordy (2013) derives GA formula for an extended version of Creditrisk+, in which LGD_i is stochastic but purely idiosyncratic. Although questionable, assuming mutual independence of X and LGD_i is a necessary step for obtaining Creditrisk+ GA in analytic form.

2.3.1 Asymptotic Capital Requirements for CreditRisk+

Since $GA = \alpha_q(L_n) - \alpha_q(E[L_n|X])$, one needs good estimations of both quantities. While the estimation of $\alpha_q(L_n)$ will be relied upon Monte Carlo simulations, $\alpha_q(E[L_n|X])$

[13] Gordy (2013) proposes regulatory calibration of ξ.

can be obtained precisely in analytic form. To do this, we first derive the expression for $E[L_n|X = x]$ and then prove that $\alpha_q(E[L_n|X]) = E[L_n|X = \alpha_q(X)]$.

Since, by definition, LGD_i and 1_{D_i} are independent, the expression for the conditionally expected portfolio loss can be obtained in the following form:

$$\begin{aligned} E[L_n|X = x] &= E[\sum_{i=1}^{n} s_i LGD_i 1_{D_i}|X = x] = \sum_{i=1}^{n} s_i E(LGD_i)E(1_{D_i}|X = x) = \\ &= \sum_{i=1}^{n} s_i E(LGD_i)\pi_i(x) = \sum_{i=1}^{n} s_i E(LGD_i)PD_i + (x-1)\sum_{i=1}^{n} s_i E(LGD_i)PD_i w_i = \\ &= A \cdot x + B, \end{aligned}$$

where A and B are constants, and $A = \sum_{i=1}^{n} s_i E(LGD_i)PD_i w_i$ and $B = \sum_{i=1}^{n} s_i E(LGD_i)PD_i - \sum_{i=1}^{n} s_i E(LGD_i)PD_i w_i$. Since $A > 0$, it follows that $E[L_n|X = x]$ is monotone increasing with respect to x. This allows us to prove the following proposition:

Proposition 2.2. *Let $\mu(X) = E[L_n|X]$ and let $L_n = \sum_{i=1}^{n} s_i LGD_i 1_{D_i}$. Then, $\alpha_q(\mu(X)) = \mu(\alpha_q(X))$.*

Proof. Since X is Gamma-distributed, x can assume any value in $(0, \infty)$. Thus, because $A > 0$, $A \cdot x + B = \mu(x)$ is a one-to-one mapping of $(0, \infty)$ onto (B, ∞). Further, observe that $B = E[L_n] - \sum_{i=1}^{n} s_i E(LGD_i)PD_i w_i$ and thus $B < E[L_n] < \alpha_q(\mu(X))$, where the second inequality holds since $\alpha_q(\mu(X)) - E[L_n]$ is a minimum capital requirement. Hence, there exists unique $x_0 \in (0, \infty)$ such that $x_0 = \mu^{-1}(\alpha_q(\mu(X)))$. Since $\mu(x)$ is monotone increasing, we have by the definition of α_q that

$$q = \Pr[\mu(X) \leq \alpha_q(\mu(X))] = \Pr[\mu^{-1}(X) \leq \mu^{-1}(\alpha_q(\mu(X)))].$$

It follows that $\mu^{-1}(\alpha_q(\mu(X))) = \alpha_q(X)$ or equivalently $\mu(\alpha_q(X)) = \alpha_q(\mu(X))$, as desired. □

2.3.2 Instrument-Level Reserve and Minimum Capital Requirements for CreditRisk+

Recall from Section 2.1 that whenever $\alpha_q(E[L_n|X]) = E[L_n|X = \alpha_q(X)]$, portfolio level asymptotic capital requirement $\mathcal{K}$ can be decomposed into instrument (exposure) level minimum capital requirements $\mathcal{K}_i$. We will see shortly that the GA formula of Gordy and Lütkebohmert (2013) is eventually expressed in terms of $\mathcal{K}_i$ and $\mathcal{R}_i$, the

latter being instrument level reserve requirement. Their derivation for CreditRisk+ is provided next.

Recall that since 1_{D_i} has Bernoulli distribution, $E[1_{D_i}|X = x] = \pi_i(x)$. The unconditional expectation of 1_{D_i} is then given by

$$E\left(E[1_{D_i}|X = x]\right) = E[\pi_i(x)] = PD_i.$$

It then follows by independence of LGD_i and 1_{D_i} that

$$\mathcal{R}_i = E[U_i] = E[LGD_i 1_{D_i}] = E[LGD_i] \cdot PD_i.$$

Instrument-level minimum capital requirements are then given by

$$\begin{aligned}\mathcal{K}_i &= E[U_i|X = \alpha_q(X)] - \mathcal{R}_i = E[LGD_i] \cdot \pi_i(\alpha_q(X)) - E[LGD_i] \cdot PD_i = \\ &= E[LGD_i] \cdot PD_i(1 + w_i(\alpha_q(X) - 1) - E[LGD_i] \cdot PD_i = \\ &= E[LGD_i] \cdot PD_i \cdot w_i(\alpha_q(X) - 1).\end{aligned}$$

These results will be used for computing $\alpha_q(E[L_n|X])$ for our simulated portfolios.

2.4 GA Formula of Gordy and Lütkebohmert (2013)

Following the derivation of Voropaev (2011), Gordy and Lütkebohmert (2013) obtain the following approximation to the true GA:

$$\alpha_q(L_n) - \mu(\alpha_q(X)) \approx -\frac{1}{2h(\alpha_q(X))} \frac{d}{dx}\left[\frac{h(x)\sigma^2(x)}{\mu'(x)}\right]\Bigg|_{x=\alpha_q(X)},$$

where $h(x)$ is a probability density function (pdf) of a systemic variable X, $\sigma^2(x) = Var[L_n|X = x]$ and $\mu(x) = E[L_n|X = x]$. Further, they derive the formula above within extended version of Single-Factor CreditRisk+ and obtain it in the following form:

$$
\begin{aligned}
GA = & -\frac{1}{2\sum_{i=1}^{n} s_i \mu_i'(\alpha_q(X))} \cdot \frac{h'(\alpha_q(X))}{h(\alpha_q(X))} \cdot \sum_{i=1}^{n} s_i^2 \\
& \times \left(\mathcal{C}_i \mu_i(\alpha_q(X)) + \mu_i^2(\alpha_q(X)) \frac{V[LGD_i]}{E[LGD_i]^2} \right) \\
& - \frac{1}{2\sum_{i=1}^{n} s_i \mu_i'(\alpha_q(X))} \cdot \sum_{i=1}^{n} s_i^2 \mu_i'(\alpha_q(X)) \\
& \times \left(\mathcal{C}_i + 2\mu_i(\alpha_q(X)) \frac{V[LGD_i]}{E[LGD_i]^2} \right),
\end{aligned} \tag{2.1}
$$

where $\mu_i(x) = E[U_i|X = x]$ and $\mathcal{C}_i = \frac{V[LGD_i]+E[LGD_i]^2}{E[LGD_i]}$. Gordy and Lütkebohmert further reparametrize (2.1) using CreditRisk+ instrument-level Reserve and Minimum Capital Requirements and show that the formula can be obtained in the following form:

$$
\begin{aligned}
& \frac{1}{2\mathcal{K}^*} \sum_{i=1}^{n} s_i^2 \left(\delta \mathcal{C}_i (\mathcal{K}_i + \mathcal{R}_i) + \delta (\mathcal{K}_i + \mathcal{R}_i)^2 \cdot \frac{V[LGD_i]}{E[LGD_i]^2} \right) - \\
& \qquad - \frac{1}{2\mathcal{K}^*} \sum_{i=1}^{n} s_i^2 \mathcal{K}_i \left(\mathcal{C}_i + 2(\mathcal{K}_i + \mathcal{R}_i) \cdot \frac{V[LGD_i]}{E[LGD_i]^2} \right),
\end{aligned} \tag{2.2}
$$

where $\mathcal{K}^* = \sum_{i=1}^{n} s_i \mathcal{K}_i$ is a minimum portfolio-level capital requirement, and $\delta = (\alpha_q(X) - 1) \cdot (\xi + \frac{1-\xi}{\alpha_q(X)})$ is a regulatory parameter. Next, the authors claim that $\delta(\mathcal{K}_i + \mathcal{R}_i)^2 \cdot \frac{V[LGD_i]}{E[LGD_i]^2}$ and $2(\mathcal{K}_i + \mathcal{R}_i) \cdot \frac{V[LGD_i]}{E[LGD_i]^2}$ are usually negligible[14] and drop these terms to obtain a simplified version of (2.2):

$$
\tilde{GA} = \frac{1}{2\mathcal{K}^*} \sum_{i=1}^{n} s_i^2 \mathcal{C}_i (\delta(\mathcal{K}_i + \mathcal{R}_i) - \mathcal{K}_i). \tag{2.3}
$$

For consistency with the IRB model, the authors porpose evaluating $\tilde{GA}$ using Basel IRB, not CreditRisk+, instrument-level Reserve and Minimum Capital Requirements. Finally, they set $\xi = 0.125$, which is the calibration of ξ that for a representative portfolio[15] matches $\tilde{GA}$ of equation (2.3) to the GA implied by CreditMetrics, a close proxy to the IRB model. Since in the regulatory setting $q = 0.999$, calibration of ξ implies calibration of δ.

[14] In fact, they evaluate these terms for six stylized portfolios and find that their values are indeed negligible relative to (2.2).

[15] Portfolio that the authors build based on their data sample on 38 German bank portfolios.

3 Setting Up Simulations

In this section we first describe how portfolios are constructed in general and then explain the generation of exposure distribution for each individual portfolio. Afterwards, we describe the Monte Carlo simulations algorithm and calibrate CreditRisk+ as well as $\tilde{GA}$ of Gordy and Lütkebohmert (2013).

3.1 Structure of Portfolios

Given the scarcity of publicly available data on bank portfolios, we decide to construct our portfolios based on the aggregate data on 38 German bank portfolios reported in (Gordy and Lütkebohmert 2013) (see Tables 2 and 3 below, copied from authors' paper).

Table 3. Granularity Adjustment for German Bank Portfolios

Portfolio	Number of Exposures	HHI	Granularity Adjustment	GA/UL
$\mathcal{P}^*$	5,289	0.0002	0.02	0.37
$\mathcal{P}$	5,289	0.0013	0.15	3.07
Large	4,000–8,999	≤ 0.0012	0.12–0.13	2.37–2.71
Medium	1,000–3,999	0.001–0.011	0.14–0.96	3.17–17.27
Small	500–999	0.004–0.011	0.36–1.14	6.57–18.65
Very Small	250–499	0.005–0.031	0.48–3.81	7.99–40.18

Notes: This table reports the GA for the reference portfolio $\mathcal{P}$ and for thirty-eight German bank portfolios with at least 250 exposures in the matched data sample. The bank portfolios were sorted by exposure count into categories labeled Large, Medium, Small, and Very Small, containing four, six, eleven, and seventeen banks, respectively. Portfolio $\mathcal{P}^*$ is a homogenized analog to reference portfolio $\mathcal{P}$ with $n = 5{,}289$ equal-sized loans and constant PD = 0.43 percent. The third column reports the range of the Herfindahl-Hirschman Index within each set of banks. The fourth column reports the corresponding range of GA calculated using equation (14), which is expressed as a percentage of total exposure. The final column reports the range of the ratio of GA to unexpected loss.

Table 2. Borrower Distribution by Rating Grade

Grade	PD Range (%)	Frequency (%)	
		Unweighted	Exposure Weighted
AAA	≤ 0.02	2.25	0.90
AA	0.02–0.06	6.07	3.63
A	0.06–0.18	23.80	18.82
BBB	0.18–1.06	60.45	66.41
BB	1.06–4.94	7.24	10.04
B	4.94–19.14	0.19	0.20
CCC	> 19.14	0.00	0.00

Notes: This table displays our mapping of borrower PD ranges into S&P whole-letter rating-grade nomenclature. The third and fourth columns report, respectively, the unweighted and exposure-weighted distribution of borrowers in reference portfolio $\mathcal{P}$.

We simulate two portfolios within each size category reported in Table 3 and set exposure count to the midpoint of the range reported in the second column. For the granularity (HHI) of portfolios within each size category, we take the lowest value and the midpoint (Column 3, Table 3) from corresponding range. This is because portfolios to which rating grade transition risk does not apply usually consist of consumer and SME loans, which are expected to be more fine-grained than corporate and mortgage portfolios[16]. Construction of portfolio exposure distribution based on HHI is discussed in Section 3.2.

PD distribution for each of the simulated portfolios is generated as follows: we shuffle the exposures (i.e. order them randomly) and allocate them across rating grades according to unweighted frequency so that AAA contains around $n * 0.0225$ exposures, AA contains around $n * 0.0607$ exposures and so on. With the number of exposures for each rating grade thus fixed, we then redistribute exposures across rating grades (if needed) so that exposure weighted frequency distribution in our simulated portfolios matches the one reported in Table 2. PD within each rating grade is taken to be the largest value of the corresponding range reported in Table 2.

In the section that follows we discuss exposure distribution generation based on HHI.

[16]Note also that for their simulations Gordy and Lütkebohmert assign a rating grade to each loan in their portfolios based on loan PD. In reality, however, many loans might not have rating grades assigned.

3.2 Exposure Distribution in Portfolios

The most straightforward way to construct exposure sizes would be to draw numbers from the actual portfolio exposure size distribution. Unfortunately, given limited data available publicly, one cannot confidently pick a distribution for this task and the resulting exposures would thus be somewhat arbitrary (not necessarily realistic). However, with respect to granularity adjustments, the loan size distribution matters only as much as it provides information about the granularity of a portfolio. Therefore, any method of construction of exposure sizes that results in a portfolio of size n with a target granularity λ would be valid for our task. We thus generate exposure ratios $\{s_i\}_{i\leq n}$ for a portfolio of size n with a target granularity λ using the following algorithm:

1. Draw n random values $\{k_i\}_{i\leq n}$ from uniform distribution with support $(0,1)$.

2. Normalize $\{k_i\}_{i\leq n}$ to 1, i.e. divide each k_i by a constant $K = \sum_{j=1}^{n} k_j$ so that $\sum_{i=1}^{n} \frac{k_i}{K} = 1$.

3. Do one of the following:

   (a) If $\sum_{i=1}^{n} (\frac{k_i}{K})^2 \approx \lambda$, set $s_i = \frac{k_i}{K}$.

   (b) If not, arrange $\{\frac{k_i}{K}\}_{i\leq n}$ in decreasing order, and set $s_i = (1-\beta_1)\frac{k_i}{K} + \frac{\beta_1}{\lfloor n\beta_2 \rfloor}$ for $i \leq \lfloor n\beta_2 \rfloor$ and $s_i = (1-\beta_1)\frac{k_i}{K}$ for $i > \lfloor n\beta_2 \rfloor$, for some $\beta_1, \beta_2 \in (0,1)$.

In words, 3(b) takes away some fraction β_1 from each $\frac{k_i}{K}$ (a total amount β_1) and evenly distributes it among $\lfloor n\beta_2 \rfloor$ largest exposure shares. Given a computational framework (e.g. a MATLAB code), β_1 and β_2 can very easily be determined by trial and error. Also, in 3(b) we have assumed that $\sum_{i=1}^{n} (\frac{k_i}{K})^2 < \lambda$. If the opposite is true, $\{\frac{k_i}{K}\}_{i\leq n}$ in 3(b) should be arragned in decreasing order instead.

It should also be noted that the simplest alternative to the algorithm above would be taking a homogeneous portfolio (which is frequently done for credit risk Monte-Carlo Simulations). However, for any given n, a homogeneous portfolio is unique and is more fine-grained than any heterogenous portfolio, as claimed in Section 2.2. The assumption of homogeneity is thus expected to significantly overstate the diversification of a portfolio. In fact, it is straightforward to see that HHI of a homogeneous portfolio with 750 exposures is 3 times smaller than HHI of the most fine-grained of eleven small portfolios ($500 \leq n < 1000$) reported in Gordy's sample.

3.3 Calibration of CreditRisk+ and $\tilde{GA}$, and the Monte Carlo Algorithm

The idea behind Credit Risk Monte Carlo simulations is to obtain N random copies of L_n and estimate $\alpha_q(L_n)$ using qth percentile in the generated sample. It is well known that the Monte Carlo estimator is efficient, i.e. it converges in probability to the true VaR with increasing sample size. In all our simulations $N = 1,000,000$. For more information on Credit Risk Monte Carlo simulations, see (Chan et al).

We run simulations on Extended Single-Factor CreditRisk+, i.e. loss outcomes are obtained using CreditRisk+ specifications. The algorithm is provided below. The calibraton is explained immediately after.

Monte Carlo Simulation Algorithm:

1. Generate a random realization x of a systemic factor X [i.e. draw a number from $\Gamma(1,1)$];

2. For each $i \in \{1,2,...,n\}$, compute $\pi_i(x)$ and run a Bernoulli trial with "success" probability $\pi_i(x)$;

3. For each $i \in \{1,2,...,n\}$, obtain LGD_i [i.e. independently draw n numbers from $Beta(1.352, 1.653)$];

4. For each $i \in \{1,2,...,n\}$, compute $s_i LGD_i 1_{D_i}$ and obtain $L_n = \sum_{i=1}^{n} s_i LGD_i 1_{D_i}$;

5. Repeat steps 1-4 N times [This generates $\{L_n^{(j)}\}_{j \leq N}$ - N independent random realizations of L_n];

6. Arrange $\{L_n^{(j)}\}_{j \leq N}$ in increasing order and return MC estimator as the$\lceil qN \rceil$th value in the ordered sample.

Based on CreditRisk+ manual, Gordy (2000) proposes setting $\xi = 1$ and $w_i = 1$ for the Single-Factor CreditRisk+. This implies that $Var[X] = 1$[17]. Since $X \sim \Gamma(\alpha, \beta)$, $E[X] = \alpha\beta = 1$ and $Var[X] = \alpha\beta^2 = 1$. Thus, $\alpha = \beta = 1$.

Since in the Extended Single-factor CreditRisk+ the distribution of LGD is not specified, we model it under $Beta$ distribution according to Moody's methodology (Moody's 2007). $E[LGD_i]$ and $V[LGD_i]$ are calibrated as in (Gordy and Lütkebohmert 2013), where $E[LGD_i] = 0.45$ by foundation IRB assumptions and $V[LGD_i] =$

[17]Note that this is also the calibration of $Var[X]$ for which CreditRisk+ and CreditMetrics produce similar loss distributions.

$0.25 \cdot E[LGD_i] \cdot (1 - E[LGD_i])$ as proposed in CP2. Further, for the *Beta* distribution with parameters α and β, the mean $\mu = \frac{\alpha}{\alpha+\beta}$ and the variance $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$. This implies $\alpha = 1.352$ and $\beta = 1.653$, i.e. $LGD_i \sim Beta(1.352, 1.653)$.

Note also that in Section 2.4 we have calibrated $\tilde{GA}$ up to $\mathcal{C}_i = \frac{V[LGD_i]+E[LGD_i]^2}{E[LGD_i]}$. Since we have assigned values to $E[LGD_i]$ and $V[LGD_i]$ in the paragraph above, the calibration of $\tilde{GA}$ follows too. Lastly, note also that both in simulations and in $\tilde{GA}$, $q = 0.999$ since the Basel IRB capital formula is derived for the 99.9th quantile.

4 Results of Simulation

Results of our simulations are reported in Table 1 below. Portfolios are ordered from least fine-grained to most fine-grained (i.e. HHI decreases from top to bottom) and GA measurements represent fractions of total portfolio value. Several observations are in order.

Firstly, the behavior of $\tilde{GA}$ is identical to that of simulated GA (GA_S): both provide higher adjustments for less diversified portfolios irrespective of size, i.e. both $\tilde{GA}$ and GA_S increase without exceptions as HHI increases. For instance, adjustments given by both GAs for a portfolio of size 375 with HHI 0.005 are smaller than adjustments for a portfolio of size 2500 with HHI 0.006. It is also noteworthy that for each of 8 portfolios, $\tilde{GA}$ is very close to HHI.

Table 1: Granularity Adjustments for 8 Simulated Portfolios

n	HHI	GA_S	$\tilde{GA}$	$GA_S - \tilde{GA}$	$\frac{GA_S - \tilde{GA}}{GA_S}$
375	0.018	0.026576	0.018781	0.007795	0.29331
750	0.0075	0.011103	0.0077883	0.0033147	0.29854
2500	0.006	0.009188	0.0061963	0.0029917	0.32561
375	0.005	0.006422	0.0055537	0.0008683	0.13521
750	0.004	0.005526	0.0042573	0.0012687	0.22959
2500	0.001	0.001296	0.0010721	0.0002239	0.17276
6500	0.0009	0.00089	0.00096966	-0.00007966	-0.0895
6500	0.0006	0.000677	0.00063152	0.00004548	0.067179

Secondly, for large, well diversified portfolios with exposure count of 6500 (last two rows) $\tilde{GA}$ approximates GA_S within 9% of GA_S. The approximation error, however, is not as good for less diversified portfolios and ranges from 13.5% to 32.5%. Moreover, the error seems to be larger with less fine-grained portfolios.

To gain more insight into the performance of GAs, we compute minimum capital requirements K^* for each portfolio using Basel IRB formula and express $\tilde{GA}$, GA_S and $GA_S - \tilde{GA}$ as a percentage of K^* (Table 2). Before comparing $\tilde{GA}$ and GA_S, observe that for poorly diversified portfolios banks might need to hold substantially more capital than suggested by Basel formula. For instance, for the least diversified portfolio in our sample (HHI of 0.018) the GA is 42% of Basel capital estimation. However, $\frac{GA_S}{K^*}$ (as well as $\frac{\tilde{GA}}{K^*}$) shrinks significantly without exceptions (to 1% eventually) as portfolios become more diversified. This is because capital requirements for each of the 8 portfolios are approximately the same, which in turn owes to the fact that K^* is not sensitive to granularity (diversification)[18] and depends on maturity[19] and portfolio quality (PD, LGD distribution) only, which are almost identical for our portfolios.

[18]Basel IRB formula is an expression that computes capital requirement K_i for each exposure under the assumption that portfolios are perfectly diversified. K^* then is simply an exposure-weighted sum of K_i, i.e. $\mathcal{K}^* = \sum_{i=1}^{n} s_i \mathcal{K}_i$.

[19]Given no data available on loan maturity, we set foundtaion IRB assumptions of 2.5 years, as do Gordy and Lütkebohmert (2013)

Table 2: Granularity Adjustments as a Fraction of Regulatory Capital

n	HHI	$\frac{GA_S}{K^*}$	$\frac{\tilde{GA}}{K^*}$	$\frac{GA_S-\tilde{GA}}{K^*}$
375	0.018	0.4238	0.29949	0.1243
750	0.0075	0.17896	0.12553	0.053427
2500	0.006	0.14847	0.10012	0.048342
375	0.005	0.098007	0.084756	0.013251
750	0.004	0.08491	0.065415	0.019494
2500	0.001	0.020235	0.016739	0.0034959
6500	0.0009	0.013772	0.015004	-0.0012326
6500	0.0006	0.01071	0.0099907	0.0007195

It is therefore natural that $\frac{GA_S-\tilde{GA}}{K^*}$ provides no surprises and behaves like $GA_S-\tilde{GA}$, i.e. it shrinks as portfolios become more diversified. For instance, we see that for poorly diversified portfolios (e.g. the one with HHI 0.018) the usage of $\tilde{GA}$ could result in an adjustment that falls short of target by the amount equal to 12% of minimum capital requirements, but for highly diversified portfolios the error $GA_S - \tilde{GA}$ could be only 7 basis points (see the last row in Table 2, for example). Thus, $\tilde{GA}$ provides reasonable estimations of GA_S for well-diversified portfolios, but this might not hold true for less diversified portfolios due to increasing approximation error that takes place with increases in HHI.

5 Conclusion

In this paper we compare the formula of Gordy and Lütkebohmert (2013) for computing regulatory GA versus the GA implied by an actuarial Extended Single-factor CreditRisk+ to see how the former performs on portfolios that carry default risk only. We simulate 8 portfolios according to exposure count, PD distribution and granularity (HHI), which are in turn chosen based on aggregate data on 38 German banks reported in the authors' paper. Further, for HHI we take conservative values from the reported range in order to reflect relatively higher diversification (lower HHI) of portfolios that carry default risk only.

After running Monte Carlo simulations, we find that the $\tilde{GA}$ of Gordy and Lütkebohmert approximates GA_S implied by Extended Single-factor CreditRisk+ within 9% of GA_S for large, well-diversified portfolios, but errs by $13.5\% - 32.5\%$ for less diversified portfolios. This calls for caution, since our results show that GA_S and $\frac{GA_S - \tilde{GA}}{GA_S}$ both increase with higher HHI, which means that for less diversified portfolios an absolute error is expected to constitute a higher fraction of a larger quantity and could thus be substantial.

In terms of regulatory implications, our findings suggest a dual approach. For portfolios with HHI below a certain benchmark value, GÃ of Gordy and Lütkebohmert could be used directly for the assessment of capital buffers under Pillar II of the Basel Accord. Less diversified portfolios, i.e. those with a higher HHI, on the other hand, require greater scrutiny. For such portfolios regulators should use a more robust approach, similar to the one employed in this paper, in order to correctly measure the amount of additionally needed regulatory capital.

6 Appendix

In this section we provide MATLAB scripts for the computational steps behind our results.

6.1 Exposure Distribution Generation Using HHI

```
MAIN SCRIPT:
----------------------
clear;
clc;

n=6500
target_HHI=0.0009
desired_error_HHI=0.01*target_HHI
printf('\n');

s0=rand(n,1);

s_s0=cumsum(s0);
sum_s0=s_s0(n);

s1=s0./sum_s0;
s1_2=s1.^2;
s_s1_2=cumsum(s1_2);
HHI_t=s_s1_2(n);



if (abs(HHI_t-target_HHI)<desired_error_HHI)
  s=s1;
else
  i=0;
  j_I=0;
  j_D=0;
```

```
  beta_1=0.3;
  beta_2=0.2;
  while(abs(HHI_t-target_HHI)>desired_error_HHI & i<1000)
    if HHI_t<target_HHI
    HHI_t=HHI_I(s1,beta_1,beta_2,n);
    s=HHI_I_s(s1,beta_1,beta_2,n);
    beta_1=beta_1+0.005;
    beta_2=beta_2-0.005;
    i=i+1;
    j_I=j_I+1;
      else
      HHI_t=HHI_D(s1,beta_1,beta_2,n);
      s=HHI_D_s(s1,beta_1,beta_2,n);
      beta_1=beta_1+0.01;
      beta_2=beta_2+0.01;
      i=i+1;
      j_D=j_D+1;
    end
  end
end

s=s';
s_s=cumsum(s);
sum_s=s_s(n);

s_2=s.^2;
s_s_2=cumsum(s_2);
sum_s_2=s_s_2(n);

save s_i.txt s;

Iterations=i
Increases=j_I
Decreases=j_D
```

```
printf('\n');

MinS=min(s)
MaxS=max(s)
Should_Be_1=sum_s
HHI_1=HHI_t
HHI_2=sum_s_2
```

SCRIPT FOR "HHI_D(L,b_1,b_2,n)":

```
function [HHI]=HHI_D(L,b_1,b_2,n)
  L_o=sort(L);
  for i=1:floor(n*b_2)
    s(i)=(1-b_1)*L_o(i)+b_1/floor(n*b_2);
  end
  for i=(floor(n*b_2)+1):n
    s(i)=(1-b_1)*L_o(i);
  end
  s_2=s.^2;
  s_s_2=cumsum(s_2);
  HHI=s_s_2(n);
end
```

SCRIPT FOR "HHI_D_s(L,b_1,b_2,n)":

```
function [s_d]=HHI_D_s(L,b_1,b_2,n)
  L_o=sort(L);
  for i=1:floor(n*b_2)
    s(i)=(1-b_1)*L_o(i)+b_1/floor(n*b_2);
  end
  for i=(floor(n*b_2)+1):n
    s(i)=(1-b_1)*L_o(i);
  end
```

```
  s_d=s;
end
```

SCRIPT FOR: "HHI_I(L,b_1,b_2,n)":

```
function [HHI]=HHI_I(L,b_1,b_2,n)
  L_o=sort(L,'descend');
  for i=1:floor(n*b_2)
    s(i)=(1-b_1)*L_o(i)+b_1/floor(n*b_2);
  end
  for i=(floor(n*b_2)+1):n
    s(i)=(1-b_1)*L_o(i);
  end
  s_2=s.^2;
  s_s_2=cumsum(s_2);
  HHI=s_s_2(n);
end
```

SCRIPT FOR "HHI_I_s(L,b_1,b_2,n)":

```
function [s_i]=HHI_I_s(L,b_1,b_2,n)
  L_o=sort(L,'descend');
  for i=1:floor(n*b_2)
    s(i)=(1-b_1)*L_o(i)+b_1/floor(n*b_2);
  end
  for i=(floor(n*b_2)+1):n
    s(i)=(1-b_1)*L_o(i);
  end
  s_i=s;
end
```

6.2 PD Distribution Generation - $[s_i\ PD_i]$ Matrix

```
clear
clc


file_name='s_6500_0.0009.txt'

s=load('-ascii', file_name);
s=s(randperm(size(s,1)),:);
size_s=size(s)
n=length(s)

for i=1:floor(0.0225*n)
  PD(i)=0.0002;
end

j=floor(0.0225*n);

for i=1:floor(0.0607*n)
  PD(j+i)=0.0006;
end

j=floor(0.0225*n)+floor(0.0607*n);

for i=1:floor(0.2380*n)
  PD(j+i)=0.0018;
end

j=j+floor(0.2380*n);

for i=1:floor(0.6045*n)
  PD(j+i)=0.0106;
end
```

```
j=j+floor(0.6045*n);

for i=1:floor(0.0724*n)
  PD(j+i)=0.0494;
end

j=j+floor(0.0724*n);

for i=1:(n-j)
  PD(j+i)=0.1914;
end

PD=PD';
size_PD=size(PD)
P=[s PD];
save sPD.txt P;

n_j=n-j
```

6.3 Computing $\alpha_q(E[L_n|X])$ of CreditRisk+

```
clear;
clc;

file_name='sPD_3_m2.txt'

ELGD=0.45;
w=1;
q=0.999;
gamma_b=1;
gamma_a=1/gamma_b;


data = load('-ascii', file_name);
n=size(data, 1)
```

```
s = reshape(data(:, 1), [n, 1]);
PD = reshape(data(:, 2), [n, 1]);


K_i=(w*ELGD*(gaminv(q,gamma_a,gamma_b)-1))*PD;
sK=s.*K_i;
sK_s=cumsum(sK);
K=sK_s(n)

sPD=s.*PD;
EL_1=ELGD*sPD;
EL_v=cumsum(EL_1);
EL=EL_v(n);
EVaR=EL+K
```

6.4 Monte Carlo Simulations

```
clear
clc

N = 1000000
q = 0.999
beta_a = 1.352
beta_b = 1.653
gamma_a = 1
gamma_b = 1
w = 1
file_name = 'sPD_0_m2.txt'
print_intermediate_values = false


data = load('-ascii', file_name);
n=size(data, 1);
```

```
s = reshape(data(:, 1), [n, 1]);
PD = reshape(data(:, 2), [n, 1]);
all_Ls = [];


% Repeat N times
for simulation = 1:N,
  % Generate a random realization x of a systemic factor X
  x = gamrnd(gamma_a, gamma_b);

  % Compute pi_i(x)
  pi = PD .* (1 + w * (x - 1));
  error_index = find(~((0 <= pi) & (pi <= 1)));
  if ~isempty(error_index),
  pi(error_index)=1;
  end

    % Run a Bernoulli trial with success probability pi_i(x)
  D = binornd(1, pi);

  % Obtain LGD_i
  LGD = betarnd(beta_a, beta_b, size(s));

  % Compute s_i * LGD_i * 1_{D_i}
  sLGD1 = s .* LGD .* D;

  % Obtain L_n
  L_v = cumsum(sLGD1);
  L = L_v(n);

  if print_intermediate_values,
    printf('\n');
    simulation
    x
    printf('i\ts\tPD\tw\tpi\tD\tLGD\tsLGD1\tL\n');
```

```
    printf('%d\t%.4f\t%.4f\t%.4f\t%.4f\t%d\t%.4f\t%.4f\t%.4f\n',
      [index s PD w pi D LGD sLGD1 L]');
  end

  % Store all computed L_n values
  all_Ls = [all_Ls; L];
end
printf('\n');

% Compute the ceil(q * N)-th largest value
all_Ls = sort(all_Ls);
VaR=all_Ls(ceil(q * N))

save Simu_0_m2.txt all_Ls;

printf('\n');
n=n
s_size=size(s)
PD_size=size(PD)
Max_LGD=max(LGD)
Min_LGD=min(LGD)
Max_PD=max(PD)
Min_PD=min(PD)
Min_L=min(all_Ls)
max_L=max(all_Ls)
Percentage_Cappings=length(error_index)/(N*n)
```

6.5 Computation of $\tilde{GA}$

MAIN SCRIPT:

```
------------
clear;
clc;

file_name = 'sPD_3_m2.txt'
```

```
data = load('-ascii', file_name);
n=size(data, 1)
s = reshape(data(:, 1), [n, 1]);
PD = reshape(data(:, 2), [n, 1]);

ELGD=0.45;
VAR_gamma=0.25;
q=0.999;

eta=0.125;
gamma_b=1/eta
gamma_a=1/gamma_b;
M=2.5

printf('\n');
GA_Vector=GA(s,PD,ELGD,VAR_gamma,q,eta,gamma_a,gamma_b,M);
GA=GA_Vector(1)
GA_Percent_K_star=GA_Vector(2)
```

SCRIPT For "GA(s,PD,ELGD,VAR_gamma,q,eta,gamma_a,gamma_b,M)":

```
function [A]=GA(s,PD,ELGD,VAR_gamma,q,eta,gamma_a,gamma_b,M)

VAR=VAR_gamma*ELGD*(1-ELGD);
C=(VAR+ELGD^2)/ELGD;
n=length(s);

for i=1:n
K(i)=Basel_IRB_K_i(PD(i),ELGD,M);
end

K=K';
```

```
sK=s.*K;
K_st=cumsum(sK);
K_star=K_st(n)

Res=ELGD*PD;

delta=(gaminv(q,gamma_a,gamma_b)-1)*(eta+(1-eta)/gaminv(q,gamma_a,gamma_b));

GA_1=C/(2*K_star);

GA_2_v=(s.^2).*(delta*(K+Res)-K);
GA_2_s=cumsum(GA_2_v);
GA_2=GA_2_s(n);

GA=GA_1*GA_2;
minK=min(K)
maxK=max(K)
Size_GA_v=size(GA_2_v);

A(1)=GA;
A(2)=GA/K_star;
end
```

```
-------------------------------------
SCRIPT For "Basel_IRB_K_i(PD(i),ELGD,M)":
-----------------------------------------
function [K]=Basel_IRB_K_i(PD,ELGD,M)

R=0.12*(1-exp(-50*PD))/(1-exp(-50))+0.24*[1-(1-exp(-50*PD))/(1-exp(-50))];

b=(0.11852-0.05478*log(PD))^2;

K=[ELGD*normcdf((1-R)^(-0.5)*norminv(PD)+(R/(1-R))^0.5*norminv(0.999))-PD*ELGD]*(1-1.5
end
```

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