Regime-Switching, Economic Disasters and "Dark Matter"

Giorgi Chavchanidze*
Boston College

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Abstract

I propose a new model of rare economic disasters. Existing models of disaster risk assume disasters are like "nuclear strikes" from which economies can only gradually recover. As a result, model-implied default-free real yield curves are downward sloping, call option prices are low, and fast recoveries - unanticipated. Existing models also face the "dark matter" problem of calibration: their quantitative implications depend on the statistical properties of time-varying disaster probability, which are quantitatively intangible due to the rarity of disasters.

I propose a regime-switching model with subjective regime beliefs[1] to address these issues. Regime-switching captures the idea that disasters are outcomes of persistent but potentially reversible bad regimes; that while disaster risk matters, so does the possibility of a boom. Subjective regime beliefs introduce a degree of freedom in the model that allows beliefs to be calibrated based on VIX. The model establishes a formal theoretical link between VIX and S&P500, can explain sudden recoveries, yield curve inversions/reversions, and time-varying discrepancy between VIX and realized volatility. The model also generates an extension of Gordon growth formula with time-varying discount rates and growth expectations, is likely to explain high put and call prices, and can offer additional quantitative insights into the puzzles already addressed by disaster risk literature.

Keywords: Regime Switching, Economic Disasters, VIX, Subjective Beliefs

*You can contact the author at chavchanidze.gio@gmail.com.

[1]The new regime-switching model is novel in that it does not depend on Markov-chain structure and arises from a new solution concept. It is a key facilitator of novel implications and can generally be applied to a wide range of models. Hence, the mathematical aspect of the paper can be considered an important contribution on its own alone.

1 Introduction

Started by Rietz (1988) and initially developed by Barro (2006), the disaster risk literature posits that economic disasters are an important psychological phenomena investors want to hedge against. It argues that modifying standard asset pricing models to adequately account for the possibility of sudden large declines in consumption can resolve many asset pricing puzzles, including the risk premium, risk free rate, excess volatility and high put option price puzzles.

Critics, however, argue that disaster risk models are not quantitatively testable, since they depend on the statistical properties of true disaster probability. These statistical properties - such as the mean, variance and persistence - are intangible, because disasters are rare and data about them - scarce (see e.g. Cochrane (2017)). For this reason, critics refer to the risk of sudden large declines in consumption as "dark matter" - it is assumed to exist and be a major driver of asset prices, but no one knows what it looks like in practice.

Apart from the "dark matter" problem, models of disaster risk face a few conceptual challenges. In particular, they struggle to explain default-free bond yield curves, volatility smiles and fast recoveries of asset prices and credit spreads. This is because in these models, disasters are like one-time nuclear strikes (or a series of low-yield nuclear strikes) - countries hit by economic disasters experience large sudden declines in consumption from which they can only gradually recover. As a result, disasters cannot be quickly "undone", expected prices of one-period risk-free assets are increasing in horizon, and the value of the index call options are less, not more, relative to the Black-Scholes log-normal world.

I propose a regime-switching model of disasters with subjective regime beliefs to address the two issues. Regimes are defined by the dynamics of aggregate consumption and dividends: in the good regime, consumption and dividends grow persistently with low volatility, while in the bad regime - they decline with high volatility. Economic disasters in this setup are then outcomes of a persistent or acute bad regimes. Regime-switching broadly captures two ideas. First, economic disasters unfold over time and events that are ex-ante expected to culminate into disasters might not necessarily do so (as a result of interventions or the nature changing its course). Second, following a disaster, the markets might anticipate a fast recovery, which could dampen the effects of disasters on asset prices. Economic disasters viewed through the lense of regime-switching are thus not as severe as "nuclear-strike" type of disasters, and the former

can provide better explanations of fast recoveries, yield curves and option prices.

The idea of modeling macroeconomic risk in the regime-switching context is of course not new and there are at least two papers (see e.g. Bhamra et al. (2010) and Andrade et al. (2023)) that employ a regime-switching setup very similar to the one in this paper. These models, however, depend on the constant probabilities of transitioning from one regime to another, whereas the current set-up completely abstracts away from transition probabilities and focuses on subjective regime likelihood in the future instead. Constant transition probabilities imply that all instances of bad regimes are ex-ante identical, and calibration based on the US data would also imply that they are mild-to-moderate. In contrast, the idea behind modeling economic disasters is that the investors care about the risk of large consumption declines and the best-performing models stipulate that such risk has time-varying intensity (see e.g. Gabaix (2012), Wachter (2013), Seo and Wachter (2019) and Ghaderi et al. (2022)).

Another alternative to the current set-up would be a regime-switching model with time-varying transition probabilities. Modeling time-variation in transition probabilities, however, faces the same quantitative challenges as modeling time-varying disaster probability. At the same time, true transition probabilities matter for asset prices only to the extent that they affect subjective regime beliefs. Thus, if one could solve a model with arbitrary beliefs about regime likelihood in the future, and had a way to reasonably calibrate subjective beliefs, transition probabilities would not matter at all. This is exactly what this paper sets out to do.

The methodological novelty of this paper is formulating and solving a regime-switching model in closed-form without transition probabilities and under minimal restrictions on subjective regime beliefs. Minimal restrictions on subjective beliefs facilitates a procedure by which regime beliefs can be backed out from VIX. VIX is in turn a crucial variable in the model, because it provides the most direct empirical measure of traders' fear of bad regimes and can be thought of as a summary statistic that aggregates many sources of information investors use in belief formation. Moreover, VIX can capture abstract pieces of information (e.g. about political risks, climate change etc) that are very difficult to represent using more primitive variables. Such information could matter for prices even if they do not always affect cash flows, since they contribute to the subjective understanding of the future macro risk.

The motivation behind backing out regime beliefs from VIX is as follows: if volatility is inherent to bad regimes, the expectations of volatility must be informative about perceived bad regime risk. Technically, regime beliefs are backed out by matching

model-implied expected volatility of S&P500 to VIX, and the procedure accounts for recent trends in volatility so that at least to a certain extent, the changes in VIX driven by transient shocks can be differentiated from the changes in VIX that reflect bad regime risk.

One potential concern with the paper could be that instead of proposing a way to mitigate the "dark matter" problem, it only suggests to replace one kind of "dark matter" with another. After all, the current formulation of the model treats subjective regime beliefs as a model primitive and is silent on its nature. What if VIX is mostly driven by animal spirits as opposed to quality information of sophisticated agents? I believe, however, that the model would mitigate the "dark matter" problem irrespective of the answer to this question. Although *subjective regime beliefs* might be more conceptually obscure than *true disaster probability*, the former is a quantifiable source of risk, while the latter is not. Hence, *true disaster probability* is a dead-end, while *subjective regime beliefs* - a potential avenue for future studies. For example, subjective regime beliefs could be rationalized in a model of Rational Inattention learning (see the attached companion paper for an early version of the model that is currently in development) in a way that is consistent with the proposed procedure for backing out regime beliefs from VIX. Alternatively, the data on regime beliefs could be analyzed from a behavioral perspective for further insights on their nature.

Although the model implications are not yet fully explored, I demonstrate some of its merits. Like models of time-varying disaster risk, time-varying subjective bad regime risk is helpful for understanding risk-free rate and risk-premium, excess volatility of stock prices relative to their dividends, and the predictability of returns by price-dividend ratios. On the side of novelty, the model can explain fast recoveries (like the one observed during Covid), default-free yield curve inversion/reversions, and time-varying discrepancy between VIX and the realized volatility. It is also likely to explain high prices of both index put and call options.

The first two novel implications arise because future outlook in the model can effect asset prices independently of current cash flows. Prices can start recovering even while cash flows are falling if future outlook recovers dramatically in the process. Yield curves can invert irrespective of current cash flows if investors start expecting things to get worse (and will revert back once they start expecting things to get better).

As for the third novel implication, the model explains the following three facts about VIX and historical volatility (HVOL): i) during good times, VIX is consistently higher than HVOL, ii) during bad times, VIX is consistently lower than HVOL, and iii) the

discrepancy is much higher during bad times. The model provides the following insights about the three facts. During good times, things can only get worse: the possibility of regime shift is accounted for by VIX, but has no effect on HVOL unless bad news arrives ("Peso Problem" phenomenon). During bad times, things can only get better: the general possibility of recovery is accounted for by VIX, but has no effect on HVOL unless recovery is in sight (Inverse "Peso Problem"). The model-implied discrepancy is higher during bad times than during good times, if the probability of recovery during bad times is higher than the probability of regime shift during good times.

I also show that the price of an index call option in a single-regime setup of the model coincides with the Black-Scholes formula generalized for a dividend-paying stock. While the generalized expression for the two-regime setup is still in the works, I argue that the resulting price should explain both high put prices and high call prices. Because good regimes create excess "upside risk" relative to the Black-Scholes world and call options insure against such risk, call prices should be higher than the standard Black-Scholes prediction. Because bad regimes create excess downside risk and put options insure against such risk, put prices should be higher than the standard Black-Scholes prediction.

The paper is organized as follows: Section 2 describes the model setup and solution, Section 3 describes the procedure for backing out regime beliefs from VIX up to 1 year, Section 4 discusses equilibrium implications and Section 5 concludes.

2 Model

2.1 General Setup

The economy is endowed with aggregate consumption C_t and non-portfolio aggregate income Y_t.[2] We are interested in pricing risk-free government bonds of different maturities and a stock market index with perpetual but stochastic dividend payout process D_t.[3] Markets are complete and infinitely-lived representative investor with time-separable log preferences prices all assets. There are two regimes: a good regime R_1 and a

[2]Including non-portfolio income in the model description allows modeling consumption and dividend processes independently and does not otherwise affect model implications.

[3]Although the insights in the paper are generally applicable to individual stocks as well, a careful treatment of stock prices requires accounting for at least two more aspects: default risk and the fact that a bad macro regime might not necessarily be bad for all individual firms. This will be subject to future studies.

bad regime R_2. Dividends D_t and aggregate consumption C_t follow a binary regime-switching diffusion process: in the good regime,

$$\frac{dC_t}{C_t} = \mu_C^{(1)} dt + \sigma_C^{(1)} dB_t,$$

$$\frac{dD_t}{D_t} = \mu_D^{(1)} dt + \sigma_D^{(1)} dB_{D,t},$$

while in the bad regime,

$$\frac{dC_t}{C_t} = \mu_C^{(2)} dt + \sigma_C^{(2)} dB_t,$$

$$\frac{dD_t}{D_t} = \mu_D^{(2)} dt + \sigma_D^{(2)} dB_{D,t},$$

$\mu_C^{(1)}, \mu_D^{(1)} > 0$, $\mu_C^{(2)}, \mu_D^{(2)} < 0$, $\sigma_C^{(1)} < \sigma_C^{(2)}$, $\sigma_D^{(1)} < \sigma_D^{(2)}$. The representative investor observes the regime the economy is in but does not know true regime switching probabilities. Instead, he has time-varying subjective beliefs $F_{t,s}$ about regime likelihood in the future, where $F_{t,s}$ denotes the time$-t$ subjective probability of being in the good regime at time $s > t$.

2.2 Subjective Regime Beliefs

Assume $F_{t,s}$ is exogenously given and is an outcome of learning about possible regime shifts in the future. With this formulation, solving the model in closed form requires the following assumptions:

Assumption 1*: There exists a finite partition of* (t, ∞) *such that* $F_{t,s} = F_t(k)$ *for all* s *inside* $k-$*th set of the partition.*

Intuitively, *Assumption 1* states that one can decompose future horizon into a finite number of disjoint segments such that the probability of being in the good regime is the same for any point within a given segment. The simplest partition is (t, ∞) itself. For this partition, $F_{t,s} = F_t \quad \forall s$. For a partition $\{(t,T), [T, \infty)\}$, $F_{t,s} = F_t(1)$ for $s \in (t,T)$ and $F_{t,s} = F_t(2)$ for $s \in [T, \infty)$. In general, the partition can vary with t without affecting model tractability, but one can assume for simplicity that it is invariant of t.

As a realistic example of the partition, one might assume that investor beliefs about being in the good regime are the same for any point within the next month but potentially differ from beliefs about the second month. Beliefs are in turn the the same for

any point within the second month but are potentially different from the beliefs about the third month and so on. The further away one goes into the future, the larger the time segments can be. For instance, investor beliefs about bad regime likelihood after 10 years would not likely be meaningfully different from their beliefs about bad regime likelihood after 11 years and so on.

Assumption 2: For any $u \in (t, s)$, $E_t [F_{u,s}] = F_{t,s}$.

Intuitively, *Assumption 2* states that the belief the representative investor expects to have tomorrow about a fixed future point s equals his belief about s today.

Assumption 3: From the perspective of agents and conditional on time t, $F_{u,u}$ is independent from C_u and D_u, $u > t$.

Assumption 3 states that the agent does not expect consumption and dividends up to time u to be predictive of his beliefs at time u about regime likelihood right after u. Intuition behind this assumption is that a bad regime is exogenous to consumption and dividends and has one-directional causal impact on them. Hence, one cannot predict a bad regime by looking at consumption and dividend realizations: if consumption and dividend processes were any different prior to subprime crisis or covid, it would not have affected the probability of the occurrence of either.

Assumption 3 is strong nevertheless relative to *Assumptions 1 and 2.* While it does not seem restrictive conditional on the economy being in the good regime in between t and u, it is somewhat strong if the economy enters a bad regime in between t and u. In this case, C_u and D_u will capture the effects of a bad regime and because in general regimes are persistent, C_u and D_u should be informative about future regime-likelihood. Assumption 3, however, is not crucial and the model can be generalized without loss of tractability by allowing beliefs to be correlated with consumption and dividends. This will be accomplished in future work.

Up to this point there is no specification of the dynamics beliefs follow. This is because the model is solved for any $F_{t,s}$ that satisfies *Assumptions 1-3.* Belief dynamics will eventually be recovered from data and Section 3 describes the methodology for backing out time-series of beliefs up to 1-year horizon.

2.3 Optimal Consumption Allocation

The representative investor solves

$$\max_{\{C_s^I\}_{s=t}^{\infty}} E_t^I \left[\int_t^{\infty} e^{-\delta(s-t)} \log C_s ds \right]$$

subject to the budget constraint

$$W_t + E_t^I \left[\int_t^{\infty} \frac{M_s}{M_t} Y_s ds \right] \geq E_t^I \left[\int_t^{\infty} \frac{M_s}{M_t} C_s^I ds \right],$$

where δ is time discounting parameter, $E_t^I[.]$ denotes time$-t$ subjective expectation of the representative investor, and M_s is a market stochastic discount factor process. W_t is time-t financial wealth, $E_t^I \left[\int_t^{\infty} \frac{M_s}{M_t} Y_s ds \right]$ is the time$-t$ present value of lifetime non-portfolio income and $E_t^I \left[\int_t^{\infty} \frac{M_s}{M_t} C_s^I ds \right]$ is the cost of consumption plan $\{C_s^I\}_{s=t}^{\infty}$. Superscript I indicates that C_s^I is a choice variable of the representative investor as opposed to time$-s$ exogenous aggregate consumption.

2.4 Equilibrium

By the first order conditions of the agent's problem (see the Appendix for details),

$$\frac{M_s}{M_t} = e^{-\delta(s-t)} \left(\frac{C_s^I}{C_t^I} \right)^{-1} \tag{2.1}$$

and by market clearing,

$$C_s^I = C_s \quad \forall s \geq t.$$

The price of a claim to time-s dividend payout D_s is thus given by

$$P_t(D_s) = E_t \left[e^{-\delta(s-t)} \left(\frac{C_s}{C_t} \right)^{-1} D_s \right]$$

and the time-t price of a risk-free government bond maturing at time s is described by

$$q_{t,s} = E_t \left[e^{-\delta(s-t)} \left(\frac{C_s}{C_t} \right)^{-1} \right].$$

We are now ready to state two main propositions of the paper.

Proposition 2.1. *(Index Price) The time-t price P_t of an index paying dividend D_s at time s satisfies*

$$P_t = \int_t^\infty P_t(D_s)ds,$$

where $P_t(D_s)$ is the time−t price of a claim to D_s and is described by

$$P_t\left(D_s\right) = e^{-(s-t)\left(\tilde{F}_{t,s}\delta_1 + \left(1-\tilde{F}_{t,s}\right)\delta_2\right)} D_t,$$

where

$$\delta_i = \delta - \left(\mu_D^{(i)} - \mu_C^{(i)} + \left(\sigma_C^{(i)}\right)^2 - \rho_{C,D}^{(i)}\sigma_C^{(i)}\sigma_D^{(i)}\right)$$

and

$$\tilde{F}_{t,s} = \sum_{i=1}^{N(s)} w_{i,s} F_t\left(i\right),$$

$\sum_{i=1}^{N(s)} w_{i,s} = 1$. *$\delta_i$ is the dividend yield in the single-regime economy perpetually in regime R_i, $N(s)$ is the number of sets in the partition of (t,∞) that intersect (t,s), $w_{i,s}$ is the length of i-th sub-partition of (t,s) divided by $t-s$, and $F_t(i)$ is the time-t subjective probability of being in the good regime at any point in time within $i-th$ segment of (t,s).*

Few examples would be illustrative. If there was only good regime R_1, $\tilde{F}_{t,s} = 1$ $\forall s > t$,

$$P_t\left(D_s\right) = e^{-(s-t)\delta_1} D_t$$

and

$$P_t = \frac{1}{\delta_1} D_t.$$

If $\tilde{F}_{t,s} = F_t \quad \forall s > t$,

$$P_t\left(D_s\right) = e^{-(s-t)(F_t\delta_1 + (1-F_t)\delta_2)} D_t$$

and

$$P_t = \frac{1}{F_t\delta_1 + \left(1-F_t\right)\delta_2} D_t.$$

In general, P_t has closed-form solution of the form

$$P_t = \frac{1}{\delta_t} D_t,$$

where δ_t increases (and the price falls) whenever perceived bad regime likelihood in-

creases at any future point in time. $\tilde{F}_{t,s}$ is an artificial probability that summarizes subjective outlook over (t,s) and is the weighted average of good regime likelihood over individual segments of (t,s). To illustrate how $\tilde{F}_{t,s}$ is constructed, consider $s=t+3$ and partition $(t,t+3)$ into $(t,t+1)$ and $[t+1,t+3)$, so that beliefs equal $F_t(1)$ for any point within $(t,t+1)$ and equal $F_t(2)$ for any point within $[t+1,t+3)$. In this example, the number $N(s)$ of segments in the partition of (t,s) equal 2, the relative length $w_{1,t+3}$ of the first segment is $\frac{1}{3}$ and the relative length $w_{2,t+3}$ of the second segment is $\frac{2}{3}$. Hence, $\tilde{F}_{t,t+3}$ is given by

$$\tilde{F}_{t,t+3} = \frac{1}{3}F_t(1) + \frac{2}{3}F_t(2).$$

Proposition 2.2. *(Bond Price) The time-t price $q_{t,s}$ of the default-free government bond paying 1 unit of consumption at time s satisfies*

$$q_{t,s} = e^{-(s-t)\left(\tilde{F}_{t,s}r_1 + \left(1-\tilde{F}_{t,s}\right)r_2\right)},$$

where $\tilde{F}_{t,s}$ is defined as in Proposition 2.1 and r_i is the risk-free rate corresponding to regime R_i and is given by

$$r_i = \delta + \mu_C^{(i)} - \left(\sigma_C^{(i)}\right)^2, \quad i=1,2.$$

Proof. Proof is in the Appendix. □

Corollary 2.1. *(Bond Yields) Time-t yield $y_{t,s}$ of a default-free government bond maturing at time s is given by*

$$y_{t,s} = \tilde{F}_{t,s}r_1 + \left(1-\tilde{F}_{t,s}\right)r_2.$$

If t and s index months, $y_{t,s}$ above is a monthly yield and $12y_{t,s}$ its annualized analogue. Because $r_1 \gg r_2$ in the data, risk-free bond yields fall and prices increase whenever subjective outlook $\tilde{F}_{t,s}$ over (t,s) worsens and vice versa.

3 Backing Out Subjective Regime Beliefs From VIX

3.1 General Idea

The section describes how to back out subjective outlook $\tilde{F}_{t,s}$ over (t,s) from VIX for $s \leq t+12$, where s and t index months. Consider $s=t+12$ and decompose 1-year

horizon into 12 1-month segments (the case for $s < t+12$) is analogous). Then, $\tilde{F}_{t,t+12}$ can be written as

$$\tilde{F}_{t,t+12} = \sum_{i=1}^{12} \frac{1}{12} F_t(i)$$

and our task is reduced to backing out $F_t(i)$ from VIX and VIX futures, $i = 1, 2, ..., 12$. The procedure for $F_t(1)$ is derived from the following equation:

$$E_t \left[\int_t^{t+1} \left(\frac{dP_s}{P_s} \right)^2 \right] = \frac{(VIX/100)^2}{12},$$

where P_s denotes time$-s$ price of S&P500, the LHS is the model expression for time$-t$ expected one-month variance of $S\&P500$, while the RHS is its VIX-based empirical analogue. Dividing VIX by 100 converts VIX from volatility point measure to standard deviation measure, raising $(VIX/100)$ to the second power converts the expected volatility measure to expected variance, while dividing $(VIX/100)^2$ by 12 converts the annualized measure into a monthly measure. Similarly, for $F_t(k)$, $k \geq 2$, we have

$$E_t \left[\int_{t+k-1}^{t+k} \left(\frac{dP_s}{P_s} \right)^2 \right] = \frac{(VIX_{t+k-1}/100)^2}{12},$$

where the LHS is model-implied time-t expected variance in between $t+k-1$ and $t+k$, while VIX_{t+k-1} is the expected value of VIX at the beginning of k-th month from t implied by VIX futures.

3.2 Exact Procedure for Current Set-Up

The current modeling design implies all changes in the expected volatility are driven by perceived bad regime risk. In this case,

$$E_t \left[\int_t^{t+1} \left(\frac{dP_s}{P_s} \right)^2 \right] = F_t(1)\,\sigma_1^2 + (1 - F_t(1))\,\sigma_2^2$$

and so $F_t(1)$ is backed out by solving the linear equation

$$F_t(1)\,\sigma_1^2 + (1 - F_t(1))\,\sigma_2^2 = \frac{(VIX/100)^2}{12},$$

where σ_i is the volatility of $S\&P500$ in regime R_i. More generally, for $k \geq 2$,

$$E_t\left[\int_{t+k-1}^{t+k}\left(\frac{dP_s}{P_s}\right)^2\right] = F_t(k)\,\sigma_1^2 + (1-F_t(k))\,\sigma_2^2$$

and so $F_t(k)$ is backed out by solving the linear equation

$$F_t(k)\,\sigma_1^2 + (1-F_t(k))\,\sigma_2^2 = \frac{(VIX_{t+k-1}/100)^2}{12}.$$

See the Appendix for detailed derivations.

3.3 Generalized Procedure

The procedure above has two main drawbacks. First, it assumes that bad regime risk is the only source of variation in VIX, but the expected volatility can also capture temporary trends that might not be related to the possibility of a regime shift. Secondly, the standard procedure is based on a setup that implies an increase in VIX must always be associated with declines in S&P500, but this is not always the case empirically. These issues are addressed by the proposition that follows:

Proposition 3.1. *Assume*[4]

$$\frac{dP_s}{P_s} = \mu_i ds + \sigma_s dB_s,$$

where

$$d\left(\sigma_s^2\right) = \bar{\kappa}\left(\bar{\theta}_i - \sigma_s^2\right)dt + \sqrt{\sigma_s^2}\bar{\sigma}_i dW_t,$$

$\bar{\theta}_i \equiv \frac{\bar{\sigma}_i^2}{4\bar{\kappa}}$, $i = 1,2$, $\bar{\theta}_1 < \bar{\theta}_2$. *Assume further that* $F_{t,s} = F_t(1)$ *for all* $s \in (t, t+1)$. *Then,*

$$E_t\left[\int_t^{t+1}\left(\frac{dP_s}{P_s}\right)^2\right] = w\sigma_t^2 + (1-w)\left(F_t(1)\,\bar{\theta}_1 + (1-F_t(1))\,\bar{\theta}_2\right),$$

$w = \frac{1}{\bar{\kappa}}\left(1-e^{-\bar{\kappa}}\right)$.

Proof. Proof is in the Appendix. □

The assumption in the proposition above states that S&P500 has stochastic mean-reverting volatility, the mean and variance of which depend on the underlying regime:

[4]The above will eventually be derived by modifying the model primitives in later work.

in the good regime, the mean and variance are low, while in the bad regime - they are high. Under this assumption, the expected volatility is determined by two terms: current trends in volatility and perceived bad regime risk. $F_t(1)$ is then backed out simply by solving the linear equation

$$w\sigma_t^2 + (1-w)\left(F_t(1)\bar{\theta}_1 + (1-F_t(1))\bar{\theta}_2\right) = \frac{(VIX/100)^2}{12},$$

where $\bar{\theta}_i$ is the mean volatility in regime R_i and the weight w is smaller the faster the volatility mean-reverts. With this result, VIX can vary through σ_t^2 even if $F_t(1)$ is unchanged, and because σ_t^2 can capture temporary effects unrelated to bad regime risk, VIX can sometimes move in the same direction as S&P500. In the good regime, σ_t^2 has small variance and VIX is not expected to move much except through changes in regime beliefs (or through changes in σ_t^2 associated with persistent shocks) . If σ_t^2 does spike temporarily because of transient shocks, it will be expected to mean-revert and will thus have limited impact on VIX.

I show in the Appendix that Proposition 3.1 can be generalized for future time periods to yield

$$E_t\left[\int_{t+k}^{t+k+1}\left(\frac{dP_s}{P_s}\right)^2\right] = wE_t\left[\sigma_{t+k}^2\right] + (1-w)\left(F_t(k+1)\bar{\theta}_1 + (1-F_t(k+1))\bar{\theta}_2\right),$$

where $E_t\left[\sigma_{t+k}^2\right]$ is a function of $\bar{\theta}_i$ and σ_t^2.

4 Equilibrium Implications

4.1 Risk-Free Rate

If we define the risk-free rate as the yield on a 1-year treasury bill, it is immediate from Section 2.4 that

$$r_t^f = \tilde{F}_{t,t+12}12r_1 + \left(1-\tilde{F}_{t,t+12}\right)12r_2,$$

where the multiplication by 12 converts a monthly yield measure to the annual measure. In the data, $12r_1 \approx 3.5\%$ and $12r_2 \approx -2\%$. Because $\tilde{F}_{t,s}$ ranges from 0 to 1, the model-implied risk-free rate would fluctuate in between 3.5% and -2% and would be lower when perceived bad regime risk is higher. For an inflation of 2%, the numbers above imply nominal rates in between 0 and 5%. If $\tilde{F}_{t,t+12}$ on average equaled 0.75, the

average real risk-free rate would be 2.1%, which is close to 2% statistic reported in Backus et al. (2011).

4.2 Risk-Premium

Recall from Section 2.4 that

$$P_t = \frac{1}{\delta_t} D_t.$$

Because δ_t is on average higher during bad regimes, while $\frac{D_{t+1}}{D_t}$ - lower, δ_t and $\frac{D_{t+1}}{D_t}$ are in general correlated. Assume, however, that the correlation occurs only across regimes, but not within, i.e.

$$\begin{aligned} Cov\left(\delta_t, \frac{D_{t+1}}{D_t}\right) &= E\left[Cov\left(\delta_t, \frac{D_{t+1}}{D_t} | R_i\right)\right] + Cov\left(E\left[\delta_t | R_i\right], E\left[\frac{D_{t+1}}{D_t} | R_i\right]\right) = \\ &= Cov\left(E\left[\delta_t | R_i\right], E\left[\frac{D_{t+1}}{D_t} | R_i\right]\right), \end{aligned}$$

where the first line holds by the Law of Total Covariance, while the second line by the assumption that $Cov\left(\delta_t, \frac{D_{t+1}}{D_t} | R_i\right) = 0$. Assume further that $E\left[\frac{\delta_t}{\delta_{t+1}} | R_i\right] = 1$, which can be motivated by mean-reverting property of δ_t (see 4.4 for intuitive discussion). Hence, the one-month return of the risky asset is given by[5]

$$\begin{aligned} r_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} &= \frac{\frac{1}{\delta_{t+1}} D_{t+1} + D_{t+1}}{\frac{1}{\delta_t} D_t} = \\ &= \left(\frac{\delta_t}{\delta_{t+1}} + \delta_t\right) \frac{D_{t+1}}{D_t}, \end{aligned}$$

where, from the Econometrician's perspective,

$$\begin{aligned} E\left[\left(\frac{\delta_t}{\delta_{t+1}} + \delta_t\right) \frac{D_{t+1}}{D_t} | R_i\right] &= E\left[\left(\frac{\delta_t}{\delta_{t+1}} + \delta_t\right) | R_i\right] E\left[\frac{D_{t+1}}{D_t} | R_i\right] \\ &= \left(1 + E\left[\delta_t | R_i\right]\right) E\left[\frac{D_{t+1}}{D_t} | R_i\right]. \end{aligned}$$

[5]More formally, because we are in continuous time set-up, $r_{t+1} = \frac{P_{t+1} + \int_t^{t+1} D_s ds}{P_t}$. However, because t indexes months, one can treat D_{t+1} as a reasonable approximation of $\int_t^{t+1} D_s ds$.

The first line follows by conditional independence of δ_t and $\frac{D_{t+1}}{D_t}$, while the second line - by $E\left[\frac{\delta_t}{\delta_{t+1}}|R_i\right]=1$. Hence, by the Law of Iterated Expectations,

$$E\left[\left(\frac{\delta_t}{\delta_{t+1}}+\delta_t\right)\frac{D_{t+1}}{D_t}\right]=p_1E\left[\left(\frac{\delta_t}{\delta_{t+1}}+\delta_t\right)\frac{D_{t+1}}{D_t}|R_1\right]+(1-p_1)E\left[\left(\frac{\delta_t}{\delta_{t+1}}+\delta_t\right)\frac{D_{t+1}}{D_t}|R_2\right]$$
$$=p_1\left(1+E\left[\delta_t|R_1\right]\right)E\left[\frac{D_{t+1}}{D_t}|R_1\right]+(1-p_1)\left(1+E\left[\delta_t|R_2\right]\right)E\left[\frac{D_{t+1}}{D_t}|R_2\right].$$

Because δ_t is increasing in subjective bad regime risk and $E\left[\delta_t|R_i\right]$ is positively related to the expected returns, bad regime risk is positively related to expected returns. Because risk-free rate is declining in bad regime risk, risk-premium is high when bad regime risk is high.

4.3 Excess Volatility

Shiller (1981) shows that the volatility of detrended S&P500 price is much higher than the volatility of its detrended dividends. Although Shiller (1981) takes this as a sign of market irrationality, I show below that the discrepancy can be rationalized in the set-up of this paper. Recall from Section 2.4 that the price of a stock market index satisfies

$$P_t=\frac{1}{\delta_t}D_t.$$

It should be obvious from this expression alone, that prices will vary excessively relative to D_t, because the former would also reflect time-varying bad regime risk through δ_t. To illustrate this formally, take some benchmark time 0 and restrict attention to time interval $(0,t)$ under the assumption that the economy remains in the good regime over this period (the case with regime-switching is more complicated but only strengthens the argument below). Recall from Section 2.1 that

$$\frac{dD_t}{D_t}=\mu_Ddt+\sigma_DdB_{D,t},$$

which implies D_t can be written as

$$D_t=D_0e^{\left(\mu_D-\frac{1}{2}\sigma_D^2\right)t+\sigma_DB_{D,t}},$$

where its detrended value is obtained as

$$\tilde{D}_t = \frac{D_t}{e^{(\mu_D - \frac{1}{2}\sigma_D^2)t}} = D_0 e^{\sigma_D B_{D,t}}.$$

Similarly, because δ_t is bounded, P_t is trending through D_t only and detrended price can be expressed as

$$\tilde{P}_t = \frac{P_t}{e^{(\mu_D - \frac{1}{2}\sigma_D^2)t}} = \frac{1}{\delta_t}\tilde{D}_t.$$

Note next that the log of detrended price is given by

$$\begin{aligned} \log \tilde{P}_t &= \log \tilde{D}_t - \log \delta_t = \\ &= \log D_0 + \sigma_D B_{D,t} - \log \delta_t, \end{aligned}$$

which implies that

$$\begin{aligned} Var\left(\log \tilde{P}_t\right) &= \sigma_D^2 t + Var\left(\log \delta_t\right) - \sigma_D Cov\left(B_{D,t}, \log \delta_t\right) > \\ &> \sigma_D^2 t = Var\left(\log \tilde{D}_t\right) \end{aligned}$$

The inequality holds, because $Cov\left(B_{D,t}, \log \delta_t\right) \leq 0$, which in turn holds as long as perceived bad regime risk is not positively correlated with dividend payouts. Hence, $\tilde{P}_t$ is more volatile than $\tilde{D}_t$, because the former also reflects the variation in bad regime risk and more generally, the uncertainty about the true data generating process of dividends. In contrast, in the single-regime economy, δ_t is constant and $Var\left(\log \tilde{P}_t\right) = Var\left(\log \tilde{D}_t\right)$.

4.4 Predictability by P/D

Predictability by P/D refers to the finding of Campbell and Shiller (1988) that price-dividend ratios can predict returns. The finding is a puzzle in standard Lucas-tree type of asset pricing models, since price dividend ratios in such models are constant. Campbell and Shiller (1988) find that price-dividend ratios are inversely related to future returns, or, equivalently, dividend yields are positively related to future returns. In the context of this paper, the finding can be understood as the consequence of time-variation in perceived bad regime risk. Investors often learn about news that have a potential to cause a regime shift, but it is difficult to differentiate ex-ante between "false alarms" and events that will actually lead to a regime shift. As a result, an

increase in perceived bad regime risk is often not associated with cash flow effects and risk perceptions revert to their average levels after the period of elevated risk is over.[6] Hence, most of the time, an increase in subjective bad regime likelihood will decrease prices without affecting cash flows. Forward dividend yields will increase as a direct consequence of this, and when a period of increased perceived risk is over and prices recover, capital gains will also be high. Hence, the positive association between current dividend yields and future returns.

I will illustrate the argument formally conditional on the economy remaining in the good regime. Start by noting that one-period gross rate of return can be expressed as[7]

$$r_{t+1} = \frac{P_{t+1}}{P_t} + \frac{D_{t+1}}{P_t} = \frac{\delta_t}{\delta_{t+1}} \frac{D_{t+1}}{D_t} + \delta_t \frac{D_{t+1}}{D_t},$$

where the first term in the sum on the RHS is the capital gain and the second term - one-month forward dividend yield. Denote $E[\delta_t | R_i] \equiv \bar{\delta}_i$ (recall that R_1 denotes the good regime) and suppose $\delta_t > \bar{\delta}_i$. Then, from the Econometrician's perspective,[8]

$$E_t[r_{t+1}|R_1] = E_t\left[\frac{\delta_t}{\delta_{t+1}}|R_1\right] E_t\left[\frac{D_{t+1}}{D_t}|R_1\right] + \delta_t E_t\left[\frac{D_{t+1}}{D_t}|R_1\right],$$

where the expected one-month future return depends positively on current dividend yield δ_t and conditional on no regime-switching taking place, δ_t is expected to mean-revert and so $E_t\left[\frac{\delta_t}{\delta_{t+1}}|R_1\right] > 1$. Similar argument applies to longer horizon returns r_{t+k}, $k > 1$.

4.5 Fast Recoveries

In existing models of slowly unfolding disaster risk (see e.g. Ghaderi et al. (2022)), cash flows effects and future outlook are tied together: future outlook is bad as long as cash flows are declining, and recovers only after cash flows recover. While the mechanism is perhaps consistent with an average recession episode, Covid pandemic demonstrated

[6] This can be seen, for instance, from the time series of VIX - VIX often spikes in response to events that have a potential to cause persistent problems, but only a few of those spikes are related to actual regime shifts.

[7] More formally, because we are in continuous time set-up, $r_{t+1} = \frac{P_{t+1} + \int_t^{t+1} D_s ds}{P_t}$. However, because t indexes months, one can treat D_{t+1} as a reasonable approximation of $\int_t^{t+1} D_s ds$.

[8] The expression assumes $\frac{\delta_t}{\delta_{t+1}}$ is independent from dividend growth conditional on R_1. See Section 4.2 for related discussion.

that this need not always be the case: following March 2020, S&P500 experienced a striking recovery that significantly preceded consumption recovery: aggregate consumption was only midway through its peak-to-trough path by the time S&P500 had recovered more than 50% of its value. This phenomenon is easy to understand in the setup of this paper, because contemporaneous cash flows and future outlook affect prices via separate channels. Recall from Section 2.4 that

$$P_t = \frac{1}{\delta_t} D_t,$$

where variation in D_t reflects cash flow effects, while variation in δ_t - changes in subjective regime likelihood in the future. Because δ_t captures many pieces of information relevant to the future economic prospects, it need not always be aligned with contemporaneous cash flow effects: investors might think that economic decline is imminent in the near future despite high current cash flows and vice versa. Hence, D_t and δ_t can push the price in different directions and whenever δ_t has stronger effect, prices can increase despite declining cash flows. This is exactly what happened during covid: prices started recovering while the economy was still in decline, because the market started anticipating fast recovery after the peak pessimism in March 2020. The anticipation of recovery is in turn evident from VIX, which declined dramatically shortly after reaching its peak in March 2020.

4.6 Yield-Curves

In this section, I show that the default-free yield curve is upward-sloping if investors expect things to get better and downward-sloping if they expect things to get worse. Moreover, yield curve will invert anytime investors become pessimistic and will revert anytime they become optimistic. If agents are pessimistic in the short-run, but optimistic in the long-run, the yield curve will be downward sloping in the short-run and upward-sloping in the long-run.

All the insights above follow from one simple observation: if s_1 and s_2 denote two future points in time, $t < s_1 < s_2$, then we have from Corollary 2.1 that

$$y_{t,s_2} - y_{t,s_1} = \left(\tilde{F}_{t,s_2} - \tilde{F}_{t,s_1}\right)(r_1 - r_2),$$

where $r_1 - r_2 > 0$ and so $y_{t,s_2} - y_{t,s_1} > 0$ iff $\tilde{F}_{t,s_2} - \tilde{F}_{t,s_1} > 0$. $\tilde{F}_{t,s_2} > \tilde{F}_{t,s_1}$ means that subjective good regime likelihood is higher over time interval (s_1, s_2) relative to (t, s_1).

Hence, if good regime likelihood is increasing in horizon, yields will also be increasing and vice versa. If $\tilde{F}_{t,s}$ is decreasing in s up to some point $\bar{s}$ and increasing afterwards, the yield curve will be downward-sloping until $\bar{s}$ and upwards-sloping afterwards. Because $\tilde{F}_{t,s}$ can change any time irrespective of cash flow realizations, yield curve can invert and revert in response to news that do not immediately affect cash flows but influence the future outlook of market participants.

4.7 Option Prices

I show in the Appendix that in the single-regime setup of the model, the price of an European index call option is given by Black-Scholes formula generalized for a dividend-paying stock. This implies that if investors were 100% sure that the economy would be in the good regime until call expiration time, the call prices would be given by the Black-Scholes formula calibrated to the non-recession data. Because a call option insures against "upside risk" and the stock upside potential is higher conditional on the economy remaining in the good regime, Black-Scholes calibrated to the non-recession data would have higher value than the Black-Scholes calibrated to average statistics (for a given stock price and strike price). Hence, for high enough subjective probability of the good regime, the model-implied call price would be higher than the Black-Scholes prediction, and the difference should be most pronounced for deeply out-of-the money calls, for which tail probabilities matter most. This could explain why difference between observed call prices and Black-Scholes prediction is increasing in the degree of out-of-moneyness.

Similar argument holds for deeply out-of-money puts, first made by Seo and Wachter (2019) in the context of Poisson jump risk. Because deeply out-of-the money puts insure against extreme downside events, they have little value in the single-regime economy (whether it is calibrated to non-recession data or average statistics). In the regime-switching setup, however, puts derive their value from the possibility of a bad regime, and this possibility matters more the greater the degree of out-of-moneyness. Hence, the Black-Scholes generalized to the regime-switching setup should be helpful for understanding the discrepancies between observed options prices and standard Black-Scholes predictions.

5 Conclusion

The paper proposes a new regime-switching model of economic disasters. It aims to address conceptual challenges of existing disaster risk models and to mitigate the "dark matter" problem of calibration in the literature. I solve the model in closed form and develop a procedure for backing out regime beliefs from VIX up to 1 year. Although quantitative implications are not yet carefully explored, I discuss several equilibrium implications and illustrate novel insights. The model is generalizable to general CRRA preferences and even less restrictive subjective regime beliefs - most likely without any loss of tractability. In future work, subjective regime beliefs can also be rationalized in a model of Rational Inattention learning, which is compatible with the VIX-based calibration and can provide a theory of belief dynamics.

6 References

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7 Appendix

7.1 Deriving the SDF

The Lagrangian of the agent's problem is given by

$$\mathcal{L}\left(\left\{C_s^I\right\}_{s=t}^{\infty}\right) = E_t^I\left[\int_t^{\infty} e^{-\delta(s-t)}\log C_s^I ds\right] + \lambda_{LM}\left(W_t + E_t^I\left[\int_t^{\infty}\frac{M_s}{M_t}Y_s ds\right] - E_t^I\left[\int_t^{\infty}\frac{M_s}{M_t}C_s^I ds\right]\right),$$

where λ_{LM} denotes a Lagrange multiplier and all other variables are defined as in Section 2.3. By standard perturbation argument, if $\{C_s^*\}$ is the optimal consumption plan, $\mathcal{L}\left(\{C_s^* + \epsilon\Delta C_s\}_{s=t}^{\infty}\right)$ is maximized at $\epsilon = 0$, i.e. $\{C_s^* + \epsilon\Delta C_s\}_{s=t}^{\infty}$ must be suboptimal for any process $\{\Delta C_s\}_{s=t}^{\infty}$ and for all values of ϵ. Hence,

$$\left.\frac{\partial\mathcal{L}\left(\{C_s^* + \epsilon\Delta C_s\}_{s=t}^{\infty}\right)}{\partial\epsilon}\right|_{\epsilon=0} = 0, \quad \forall\{\Delta C_s\}$$

which implies

$$E_t^I\left[\int_t^{\infty}\left(e^{-\delta(s-t)}\frac{1}{C_s^*} - \lambda_{LM}\frac{M_s}{M_t}\right)\Delta C_s ds\right] = 0, \quad \forall\{\Delta C_s\}.$$

The above can hold $\forall\{\Delta C_s\}$ iff the integrand is identically zero. Hence,

$$e^{-\delta(s-t)}\frac{1}{C_s^*} = \lambda_{LM}\frac{M_s}{M_t} \quad \forall s \geq t,$$

which implies

$$\frac{1}{C_t^*} = \lambda_{LM}$$

and

$$\frac{M_s}{M_t} = e^{-\delta(s-t)}\frac{C_t^*}{C_s^*}.$$

7.2 Auxiliary Derivations

Lemmas proved in this section are used later in the derivation of main results.

Lemma 7.1. *Suppose*

$$\frac{dC_t}{C_t} = \mu_C dt + \sigma_C dB_t,$$

r denotes the risk-free rate and the SDF process M_t is given by

$$M_t \equiv e^{-\delta t}\left(\frac{C_t}{C_0}\right)^{-1}.$$

Then, the following are true:

(1) $\frac{dM_t}{M_t} = -\left(\delta + \mu_C - \sigma_C^2\right)dt - \sigma_C dB_t$

(2) $\log M_t = -\left(\delta + \mu_C - \frac{1}{2}\sigma_C^2\right)t - \sigma_C B_t$

(3) $r = \delta + \mu_C - \sigma_C^2$

(4) $\log \frac{M_T}{M_t} = -\left(r + \frac{1}{2}\sigma_C^2\right)\tau - \sigma_C\sqrt{\tau}Z_{C,\tau}, \quad Z_{C,\tau} \sim N(0,1), \quad \tau \equiv T - t.$

Proof. (1) follows by straightforward application of Ito's Lemma:

$$dM_t = -e^{-\delta t}\left(\frac{C_t}{C_0}\right)^{-1}\delta dt - e^{-\delta t}\left(\frac{C_t}{C_0}\right)^{-1}\frac{dC_t}{C_t} + e^{-\delta t}\left(\frac{C_t}{C_0}\right)^{-1}\left(\frac{dC_t}{C_t}\right)^2 =$$

$$= -M_t\delta dt - M_t\frac{dC_t}{C_t} + M_t\left(\frac{dC_t}{C_t}\right)^2,$$

which implies

$$\frac{dM_t}{M_t} = -\delta dt - \mu_C dt - \sigma_C dB_t + \sigma_C^2 dt = -\left(\delta + \mu_C - \sigma_C^2\right)dt - \sigma_C dB_t.$$

(2) Follows directly from *(1)*: dynamics of $\frac{dM_t}{M_t}$ imply

$$M_t = M_0 \exp\left(-\left(\delta + \mu_C - \sigma_C^2\right)t - \frac{1}{2}\sigma_C^2 t - \sigma_C B_t\right)$$

and since $M_0 = 1$ by the definition of an SDF process, it is immediate that

$$\log M_t = -\left(\delta + \mu_C - \frac{1}{2}\sigma_C^2\right)t - \sigma_C B_t.$$

To see *(3)*, let R_t denote the return on the risk-free asset. By definition of an SDF process, $M_t R_t$ must be a martingale, i.e. $d(M_t R_t)$ must have a zero drift. By Ito's Lemma,

$$d(M_t R_t) = R_t dM_t + M_t dR_t = M_t R_t\frac{dM_t}{M_t} + M_t R_t\frac{dR_t}{R_t}$$

and so

$$\frac{d(M_t R_t)}{M_t R_t} = \frac{dM_t}{M_t} + \frac{dR_t}{R_t} = \frac{dM_t}{M_t} + r_t dt.$$

Because $d(M_t R_t)$ has zero drift iff $\frac{d(M_t R_t)}{M_t R_t}$ has zero drift, it follows

$$E\left[\frac{dM_t}{M_t}\right] = -r_t dt$$

and we get by *(1)* that

$$r_t = \delta + \mu_C - \sigma_C^2 \equiv r.$$

To prove *(4)* consider $\log M_s$ as a function of M_s and note by Ito's Lemma that

$$d\log M_s = \frac{dM_s}{M_s} - \frac{1}{2}\left(\frac{dM_s}{M_s}\right)^2 =$$

$$= -\left(r + \frac{1}{2}\sigma_C^2\right)dt - \sigma_C dB_t.$$

Hence, setting $\tau \equiv T - t$, we have

$$\int_t^T d\log M_s = -\int_t^T \left(r + \frac{1}{2}\sigma_C^2\right)dt - \int_t^T \sigma_C dB_t =$$

$$= -\left(r + \frac{1}{2}\sigma_C^2\right)\tau - \sigma_C\left(B_T - B_t\right),$$

where $B_T - B_t \sim N(0, \tau)$ and so

$$Z_{C,\tau} \equiv \frac{1}{\sqrt{\tau}}\left(B_T - B_t\right) \sim N(0,1).$$

Finally, noting that

$$\int_t^T d\log M_s = \log M_T - \log M_t = \log\left(\frac{M_T}{M_t}\right),$$

we finally get

$$\log\left(\frac{M_T}{M_t}\right) = -\left(r + \frac{1}{2}\sigma_C^2\right)\tau - \sigma_C\sqrt{\tau}Z_{C,\tau}.$$

□

Lemma 7.2. *Suppose*

$$\frac{dC_t}{C_t} = \mu_C dt + \sigma_C dB_t,$$

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dB_{D,t}$$

and set

$$M_t \equiv e^{-\delta t}\left(\frac{C_t}{C_0}\right)^{-1}$$

$$P_t \equiv E_t\left[\int_t^\infty \frac{M_s}{M_t} D_s dt\right].$$

Define

$$\mu_{D/C} \equiv \mu_D - \mu_C + \sigma_C^2 - \rho_{C,D}\sigma_C\sigma_D,$$

$$\sigma_{D/C} \equiv \sqrt{\sigma_D^2 + \sigma_C^2 - 2\rho_{C,D}\sigma_C\sigma_D}.$$

Then, the following are true:

(1) $\frac{d(D_t/C_t)}{D_t/C_t} = \mu_{D/C}dt + \sigma_{D/C}dB_{D/C,t}$,
(2) $P_t = \frac{D_t}{\delta - \mu_{D/C}}$
(3) $dP_t = dD_t$
(4) $\log\frac{P_T}{P_t} = \left(\mu_D - \frac{1}{2}\sigma_D^2\right)\tau + \sigma_D\sqrt{\tau}Z_{D,\tau}, \quad Z_{D,\tau} \sim N(0,1), \quad \tau \equiv T - t.$

Proof. To see *(1)*, start by noting that Ito's Lemma implies

$$d(D_s/C_s) = -\frac{D_s}{C_s^2}dC_s + \frac{1}{C_s}dD_s + \frac{D_s}{C_s^3}dC_s^2 - \frac{1}{C_s^2}dC_s dD_s =$$

$$= -\frac{D_s}{C_s}\frac{dC_s}{C_s} + \frac{D_s}{C_s}\frac{dD_s}{D_s} + \frac{D_s}{C_s}\frac{dC_s^2}{C_s^2} - \frac{D_s}{C_s}\frac{dC_s}{C_s}\frac{dD_s}{D_s}$$

and so

$$\frac{d(D_s/C_s)}{D_s/C_s} = -\frac{dC_s}{C_s} + \frac{dD_s}{D_s} + \left(\frac{dC_s}{C_s}\right)^2 - \frac{dC_s}{C_s}\frac{dD_s}{D_s} =$$

$$= \left(\mu_D - \mu_C + \sigma_C^2 - \rho_{C,D}\sigma_C\sigma_D\right)dt + \sigma_D dB_{D,s} - \sigma_C dB_s.$$

Define

$$dB_{D/C,s} \equiv \frac{\sigma_D dB_{D,s} - \sigma_C dB_s}{\sqrt{\sigma_D^2 + \sigma_C^2 - 2\rho_{C,D}\sigma_C\sigma_D}}$$

and note that

$$E\left[dB_{D/C}\right] = 0,$$

$$\left(dB_{D/C,s}\right)^2 = \frac{\sigma_D^2 dt + \sigma_C^2 dt - 2\rho_{C,D}\sigma_D\sigma_C dt}{\sigma_D^2 + \sigma_C^2 - 2\rho_{C,D}\sigma_C\sigma_D} = dt,$$

and

$$\int_0^t \left(dB_{D/C}\right)^2 = t.$$

Because $dB_{D/C}$ has no drift, it is a continuous-time martingale with quadratic variation equal to t, which implies by Levy Theorem that $B_{D/C}$ is a Brownian Motion. Hence, $\frac{d(D_s/C_s)}{D_s/C_s}$ follows standard Brownian drift-diffusion process given by

$$\frac{d(D_s/C_s)}{D_s/C_s} = \mu_{D/C}dt + \sigma_{D/C}dB_{D/C},$$

$$\mu_{D/C} \equiv \mu_D - \mu_C + \sigma_C^2 - \rho_{C,D}\sigma_C\sigma_D,$$
$$\sigma_{D/C} \equiv \sqrt{\sigma_D^2 + \sigma_C^2 - 2\rho_{C,D}\sigma_C\sigma_D}.$$

To see *(2)*, start by noting that

$$P_t = E_t\left[\int_t^\infty e^{-\delta(s-t)}\frac{C_t}{C_s}D_s ds\right] = C_t\int_t^\infty e^{-\delta(s-t)}E_t\left[\frac{D_s}{C_s}\right]ds,$$

where

$$\frac{D_s}{C_s} = \frac{D_t}{C_t}\exp\left(\mu_{D/C}(s-t) - \frac{1}{2}\sigma_{D/C}^2(s-t) + \sigma_{D/C}\left(B_{D/C,s} - B_{D/C,t}\right)\right)$$

by *(1)*. Because $B_{D/C,s} - B_{D/C,t} \sim N(0, s-t)$,

$$E_t\left[e^{\sigma_{D/C}\left(B_{D/C,s} - B_{D/C,t}\right)}\right] = e^{\frac{1}{2}\sigma_{D/C}^2(s-t)}$$

and

$$E_t\left[\frac{D_s}{C_s}\right] = \frac{D_t}{C_t}e^{\mu_{D/C}(s-t)}, \tag{7.1}$$

which in turn implies

$$P_t = D_t\frac{1}{\mu_{D/C} - \delta}\int_t^\infty e^{\left(\mu_{D/C}-\delta\right)(s-t)}d\left(\left(\mu_{D/C} - \delta\right)(s-t)\right) =$$

$$= \frac{D_t}{\delta - \mu_{D/C}}.$$

(3) follows from *(2)* by Ito's Lemma:

$$dP_t = \frac{1}{\delta - \mu_{D/C}}dD_t = \frac{D_t}{\delta - \mu_{D/C}}\frac{dD_t}{D_t} = P_t\frac{dD_t}{D_t}.$$

(4) follows from *(3)* and derivation that is perfectly analogous to part *(4)* of Lemma

7.1. □

Lemma 7.3. *Let X_1 and X_2 denote two jointly Normal random variables with means μ_1 and μ_2, variances σ_1 and σ_2, correlation coefficient ρ, and covariance σ_{12}. Then,*

$$E\left[e^{X_1+X_2}|X_1>a\right]=e^{\mu_1+\mu_2+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_1^2+\sigma_{12}}N\left(\frac{\mu_1+\sigma_1^2+\sigma_{12}-a}{\sigma_1}\right),$$

where $N(.)$ is the cumulative density function of Standard Normal.

Proof. It is well-known that the joint probability density function (pdf) of X_1 and X_2 can be obtained as

$$f_{X_1,X_2}(x_1,x_2)=\frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}\times\frac{1}{\sqrt{2\pi}\left(\sigma_2\sqrt{1-\rho^2}\right)}e^{-\frac{1}{2}\left(\frac{x_2-b_{x_1}}{\sigma_2\sqrt{1-\rho^2}}\right)^2}\equiv$$

$$\equiv f_{X_1}(x_1)\times\tilde{f}(x_1,x_2),$$

where $b_{x_1}\equiv\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)$, $f_{X_1}(x_1)$ is a pdf of $N(\mu_1,\sigma_1)$ and $\tilde{f}(x_1,x_2)$ is a pdf of $N\left(b_{x_1},\sigma_2\sqrt{1-\rho^2}\right)$[9]. Hence,

$$E\left[e^{X_1+X_2}|X_1>a\right]=\int_a^\infty\int_{-\infty}^\infty e^{x_1+x_2}f_{X_1}(x_1)\times\tilde{f}(x_1,x_2)\,dx_2dx_1=$$

$$=\int_a^\infty e^{x_1}f_{X_1}(x_1)\int_{-\infty}^\infty e^{x_2}\tilde{f}(x_1,x_2)\,dx_2dx_1=$$

$$=\int_a^\infty e^{x_1}f_{X_1}(x_1)\,e^{b_{x_1}+\frac{1}{2}\sigma_2^2\left(1-\rho^2\right)}dx_1, \tag{7.2}$$

where the last equality follows from the fact that $\int_{-\infty}^\infty e^{x_2}\tilde{f}(x_1,x_2)\,dx_2$ is a moment-generating function of $N\left(b_{x_1},\sigma_2\sqrt{1-\rho^2}\right)$ evaluated at $t=1$. Next, denote $X_1\equiv\mu_1+\sigma_1Z_1$, $Z_1\sim N(0,1)$, and note the following equalities:

$$dx_1=\sigma_1dz_1,$$

$$b_{x_1}=\mu_2-\rho\frac{\sigma_2}{\sigma_1}\mu_1+\rho\frac{\sigma_2}{\sigma_1}(\mu_1+\sigma_1z_1)=\mu_2+\rho\sigma_2z_1,$$

[9]See https://www.sciencedirect.com/topics/mathematics/bivariate-normal-distribution for a detailed derivation.

and

$$x_1 + b_{x_1} + \frac{1}{2}\sigma_2^2\left(1-\rho^2\right) = \mu_1 + \mu_2 + \frac{1}{2}\sigma_2^2 - \frac{1}{2}\sigma_2^2\rho^2 + \left(\sigma_1 + \rho\sigma_2\right) z_1.$$

Hence,

$$\int_a^\infty e^{x_1} f_{X_1}(x_1)\, e^{b_{x_1} + \frac{1}{2}\sigma_2^2\left(1-\rho^2\right)} dx_1 =$$

$$= e^{\mu_1+\mu_2+\frac{1}{2}\sigma_2^2-\frac{1}{2}\sigma_2^2\rho^2} \int_{\frac{a-\mu_1}{\sigma_1}}^\infty e^{(\sigma_1+\rho\sigma_2)z_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_1)^2} dz_1, \tag{7.3}$$

where

$$-\frac{1}{2}(z_1)^2 + (\sigma_1 + \rho\sigma_2) z_1 = -\frac{1}{2}\left(z_1 - (\sigma_1 + \rho\sigma_2)\right)^2 + \frac{1}{2}(\sigma_1 + \rho\sigma_2)^2$$

and so

$$\int_{\frac{a-\mu_1}{\sigma_1}}^\infty e^{(\sigma_1+\rho\sigma_2)z_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_1)^2} dz_1 =$$

$$= e^{\frac{1}{2}(\sigma_1+\rho\sigma_2)^2} \int_{\frac{a-\mu_1}{\sigma_1}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_1-(\sigma_1+\rho\sigma_2))^2} dz_1 =$$

$$= e^{\frac{1}{2}(\sigma_1+\rho\sigma_2)^2} N\left(\frac{\mu_1 + \sigma_1^2 + \sigma_{12} - a}{\sigma_1}\right). \tag{7.4}$$

Putting (7.2), (7.3) and (7.4) together, we finally get

$$E\left[e^{X_1+X_2} | X_1 > a\right] = e^{\mu_1+\mu_2+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_1^2+\sigma_{12}} N\left(\frac{\mu_1 + \sigma_1^2 + \sigma_{12} - a}{\sigma_1}\right).$$

□

Corollary 7.1. *Define X_1 and X_2 as in Lemma 7.3. Then,*

$$E\left[e^{X_2+a} | X_1 > a\right] = e^{a+\mu_2+\frac{1}{2}\sigma_2^2} N\left(\frac{\mu_1 + \sigma_{12} - a}{\sigma_1}\right).$$

Proof. Follows by analogous derivation. □

7.3 Call Option Price in a Single-Regime Economy

Let S_t be the price of a stock with underlying dividend process given by

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dB_{D,t}.$$

Let H_t denote the time−t price of the European call option written on this stock and let K and T denote the strike price and the maturity date of the call option, $\tau \equiv T - t$.

Then,

$$H_t^{Call} = S_t e^{-\delta_D \tau} N(d_1) - K e^{-r\tau} N(d_2),$$

$$d_1 = \frac{\left(r - \delta_D + \frac{1}{2}\sigma_S^2\right)\tau + \log\left(\frac{S_t}{K}\right)}{\sigma_S \sqrt{\tau}},$$

$$d_2 = d_1 - \sigma_S \sqrt{\tau},$$

where σ_S is the volatility of S_t process and δ_D is the dividend yield of the underlying stock.

Proof. Start by noting that the price of the call option with strike price K and expiration date T is given by

$$H_t^{Call} = E_t\left[\frac{M_T}{M_t}(S_T - K)^+\right].$$

Because

$$\frac{M_T}{M_t} = e^{\log \frac{M_T}{M_t}}$$

and

$$(S_T - K)^+ = S_t\left(\frac{S_T}{S_t} - \frac{K}{S_t}\right)^+ = S_t\left(e^{\log\left(\frac{S_T}{S_t}\right)} - \frac{K}{S_t}\right)^+,$$

H_t can be written as

$$H_t = S_t E_t\left[e^{\log \frac{M_T}{M_t}}\left(e^{\log\left(\frac{S_T}{S_t}\right)} - \frac{K}{S_t}\right)^+\right].$$

By the Law of Iterated Expectations,

$$H_t = S_t E_t\left[e^{\log \frac{M_T}{M_t}}\left(e^{\log\left(\frac{S_T}{S_t}\right)} - \frac{K}{S_t}\right)\middle|\log\left(\frac{S_T}{S_t}\right) > \log\left(\frac{K}{S_t}\right)\right]$$

and by Lemmas 7.2 and 7.1,

$$\log\frac{S_T}{S_t} = \left(\mu_D - \frac{1}{2}\sigma_D^2\right)\tau + \sigma_D\sqrt{\tau}Z_{D,\tau},$$

$$\log\frac{M_T}{M_t} = -\left(r + \frac{1}{2}\sigma_C^2\right)\tau - \sigma_C\sqrt{\tau}Z_{C,\tau},$$

$$Corr\left(Z_{D,\tau}, Z_{C,\tau}\right) = \rho. \tag{7.5}$$

Because $\log \frac{S_T}{S_t}$ and $\log \frac{M_T}{M_t}$ are both Normally distributed, denote

$$\log \frac{S_T}{S_t} \equiv X_1 \sim N(\mu_1, \sigma_1), \tag{7.6}$$

$$\log \frac{M_T}{M_t} \equiv X_2 \sim N(\mu_2, \sigma_2), \tag{7.7}$$

$$\sigma_{12} \equiv Cov(X_1, X_2),$$

$$\log\left(\frac{K}{S_t}\right) \equiv a$$

and rewrite H_t as

$$H_t = S_t E_t\left[e^{X_2}\left(e^{X_1} - e^a\right) | X_1 > a\right] =$$

$$= S_t\left(E_t\left[e^{X_2+X_1} | X_1 > a\right] - E_t\left[e^{X_2+a} | X_1 > a\right]\right). \tag{7.8}$$

By Lemma 7.3 and Corollary 7.1**,**

$$E_t\left[e^{X_2+X_1} | X_1 > a\right] = e^{\mu_1+\mu_2+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2+\sigma_{12}} N(d_1), \tag{7.9}$$

$$E_t\left[e^{X_2+a} | X_1 > a\right] = e^{a+\mu_2+\frac{1}{2}\sigma_2^2} N(d_2), \tag{7.10}$$

$$d_1 \equiv \frac{\mu_1 + \sigma_1^2 + \sigma_{12} - a}{\sigma_1},$$

$$d_2 \equiv \frac{\mu_1 + \sigma_{12} - a}{\sigma_1} = d_1 - \sigma_1,$$

$N(.)$ Standard Normal cdf, and by (7.6), (7.7) and (7.5),

$$\mu_1 = \left(\mu_D - \frac{1}{2}\sigma_D^2\right)\tau,$$

$$\sigma_1 = \sigma_D\sqrt{\tau},$$

$$\mu_2 = -\left(\delta + \mu_C - \frac{1}{2}\sigma_C^2\right)\tau,$$

$$\sigma_2 = \sigma_C\sqrt{\tau},$$

$$\sigma_{12} = -\rho\sigma_C\sigma_D\tau.$$

Hence,

$$\mu_2 + \frac{1}{2}\sigma_2^2 = -\left(\delta + \mu_C - \sigma_C^2\right)\tau = -r\tau,$$

$$\mu_1 + \mu_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \sigma_{12} = \left(-\delta + \mu_D - \mu_C + \sigma_C^2 - \rho\sigma_C\sigma_D\right)\tau = -\delta_D\tau,$$

where r is the risk-free rate, δ_D is the dividend yield of the underlying stock, the last equality on the first line follows by Part 3 of Lemma 7.1**,** and the last equality on the second line - by Part 2 of Lemma 7.2. Similarly,

$$d_2 = d_1 - \sigma_D\sqrt{\tau} \tag{7.11}$$

and because

$$a \equiv \log\left(\frac{K}{S_t}\right) = -\log\left(\frac{S_t}{K}\right)$$

and

$$r - \delta_D = \mu_D - \rho\sigma_C\sigma_D,$$

we have

$$d_1 = \frac{\mu_1 + \sigma_1^2 + \sigma_{12} - a}{\sigma_1} = \frac{\left(\mu_D + \frac{1}{2}\sigma_D^2 - \rho\sigma_C\sigma_D\right)\tau - a}{\sigma_D\sqrt{\tau}} =$$

$$= \frac{\left(r - \delta_D + \frac{1}{2}\sigma_S^2\right)\tau + \log\left(\frac{S_t}{K}\right)}{\sigma_S\sqrt{\tau}}, \tag{7.12}$$

where σ_S is the volatility of $\frac{dS_t}{S_t}$ and equals σ_D by Part 3 of Lemma 7.2**.** Inserting these back into (7.9) and (7.10), we have by (7.8) that

$$H_t = S_t e^{-\delta_D\tau} N\left(d_1\right) - K e^{-r\tau} N\left(d_2\right),$$

which, along with (7.11) and (7.12), is exactly the Black-Scholes formula generalized for a dividend-paying stock with constant dividend yield. □

7.4 Proofs of Main Propositions

7.4.1 Proposition 2.1

Unless otherwise stated, assume Assumptions 1-3 in Section 2.2 hold. Proofs in this section proceed (and build on each other) in the following order:

1. Price a claim to a dividend payout at a future point in time in a single-regime

economy.

2. Price a time$-t$ claim to a dividend payout D_s at a future point s in a restricted regime-switching economy with any regime beliefs. The restriction is as follows: decompose (t, s) into disjoint segments of equal length Δ and assume regimes are determined at the start point of each segment and persist for sure until the end-point of the same segment. Call the resulting expression $P_t(D_s, \Delta)$.

3. Price a time$-t$ claim to a dividend payout D_s at a future point s in an unrestricted regime-switching economy but with restricted beliefs. The restriction on beliefs is as follows: conditional on time t, subjective likelihood of being in the good regime is the same for all future points $s > t$ (but can vary as t varies). Regime-switching is unrestricted in the sense that the price is derived in the limit as $\Delta \to 0$. In the limit, regimes can switch at any point in time and persist for sure over arbitrarily small time interval only.

4. Generalize pricing equation in 3. above by allowing beliefs to be arbitrary under Assumption 1 in Section 2.2.

Lemma 7.4. *Suppose the economy is in regime R_i at some time $\tilde{t}$ and that the regime will persist for sure until some time $\tilde{s} > \tilde{t}$. Then, time-$\tilde{t}$ value $\tilde{P}_{\tilde{t}}(D_{\tilde{s}})$ of a claim to dividend payout $D_{\tilde{s}}$ in time $\tilde{s}$ equals*

$$\tilde{P}_{\tilde{t}}(D_{\tilde{s}}) = D_{\tilde{t}} e^{-(\tilde{s}-\tilde{t})\delta_i},$$

where

$$\delta_i = \delta - \left(\mu_D^{(i)} - \mu_C^{(i)} + \left(\sigma_C^{(i)}\right)^2 - \rho_{C,D}^{(i)} \sigma_C^{(i)} \sigma_D^{(i)} \right).$$

Proof. Denote by $E_{\tilde{t}}^i[.]$ time-$\tilde{t}$ conditional expectation in a single-regime-R_i economy and note the following chain of equalities:

$$\tilde{P}_{\tilde{t}}(D_{\tilde{s}}) = E_{\tilde{t}}^i \left[e^{-(\tilde{s}-\tilde{t})\delta} \frac{C_{\tilde{t}}}{C_{\tilde{s}}} D_{\tilde{s}} \right] = C_{\tilde{t}} e^{-(\tilde{s}-\tilde{t})\delta} E_{\tilde{t}}^i \left[\frac{D_{\tilde{s}}}{C_{\tilde{s}}} \right] =$$

$$= D_{\tilde{t}} e^{-(\tilde{s}-\tilde{t})\left(\delta - \mu_{D/C}^{(i)}\right)} = D_{\tilde{t}} e^{-(\tilde{s}-\tilde{t})(\delta_i)}.$$

The first equation holds by Section 7.1, the second is trivial, while the third and the fourth follow by (7.1) in the proof of Lemma 7.2. □

Lemma 7.5. *Take arbitrary $s > t$ and partition $(t, t+s)$ into intervals of equal width Δ. Assume that a regime can switch only at times $s-(k+1)\Delta$, $k = 0, 1, 2, ..., \frac{s-t-\Delta}{\Delta}$, and once the regime is determined at time $s-(k+1)\Delta$, it persists for sure until $s-k\Delta$. Let $F_{t,s-(k+1)\Delta}$ denote the time−t probability of being in the good regime (R_1) at time $s-(k+1)\Delta$ and let $P_t(D_s, \Delta)$ denote the time-t price of a claim to time-s dividend payout D_s for fixed Δ. Then,*

$$P_t(D_s, \Delta) = \prod_{k=0}^{\frac{s-t}{\Delta}-1} \left(F_{t,s-(k+1)\Delta} e^{-\Delta\delta_1} + \left(1 - F_{t,s-(k+1)\Delta}\right) e^{-\Delta\delta_2} \right) D_t.$$

Proof. Let $R_{s-(k+1)\Delta}$ be a binary variable that equals R_1 if a good regime is drawn at time $s-(k+1)\Delta$ and that equals R_2 otherwise. Further, denote by $E_{s-(k+1)\Delta}[\cdot]$ expectation operator conditional on all available information up to $s-(k+1)\Delta$ but excluding $R_{s-(k+1)\Delta}$, and let $F_{s-(k+1)\Delta,s-(k+1)\Delta}$ denote time $s-(k+1)\Delta$ subjective probability that a good regime will be drawn at time $s-(k+1)\Delta$.[10] Then, the price of a claim to D_s right before $s-\Delta$ is derived by the following chain of equalities:

$$P_{s-\Delta}(D_s, \Delta) = E_{s-\Delta}\left[e^{-(\Delta)\delta}\frac{C_{s-\Delta}}{C_s}D_s\right] = E_{s-\Delta}\left[E_{s-\Delta}\left[e^{-(\Delta)\delta}\frac{C_{s-\Delta}}{C_s}D_s\,|R_{s-\Delta}\right]\right] = \tag{7.13}$$

$$= F_{s-\Delta,s-\Delta}E_{s-\Delta}\left[e^{-(\Delta)\delta}\frac{C_{s-\Delta}}{C_s}D_s\,|R_{s-\Delta} = R_1\right] +$$

$$+\left(1 - F_{s-\Delta,s-\Delta}\right)E_{s-\Delta}\left[e^{-(\Delta)\delta}\frac{C_{s-\Delta}}{C_s}D_s\,|R_{s-\Delta} = R_2\right] = \tag{7.14}$$

$$= F_{s-\Delta,s-\Delta}D_{s-\Delta}e^{-\Delta\delta_1} + \left(1 - F_{s-\Delta,s-\Delta}\right)D_{s-\Delta}e^{-\Delta\delta_2} = \tag{7.15}$$

$$= \left(F_{s-\Delta,s-\Delta}e^{-\Delta\delta_1} + \left(1 - F_{s-\Delta,s-\Delta}\right)e^{-\Delta\delta_2}\right)D_{s-\Delta}. \tag{7.16}$$

The first line in the equations above follows by the law of iterated expectations, the second line - by expanding the outer expectation, third line - by Lemma 7.4 when $\tilde{s}$ is replaced by s and $\tilde{t}$ by $s-\Delta$. The last line is trivial. Next, note by the law of iterated expectations that

$$P_t(D_s, \Delta) = E_t\left[e^{-(s-t)\delta}\frac{C_t}{C_s}D_s\right] = E_t\left[E_{s-\Delta}\left[e^{-(s-t)\delta}\frac{C_t}{C_s}D_s\right]\right] = \tag{7.17}$$

[10] Assume regimes are drawn at times $s-(k+1)\Delta+\epsilon$ for small ϵ.

$$= E_t\left[e^{-(s-\Delta-t)\delta}\frac{C_t}{C_{s-\Delta}}E_{s-\Delta}\left[e^{-(\Delta)\delta}\frac{C_{s-\Delta}}{C_s}D_s\right]\right] = \tag{7.18}$$

$$= E_t\left[e^{-(s-\Delta-t)\delta}\frac{C_t}{C_{s-\Delta}}P_{s-\Delta}\left(D_s,\Delta\right)\right]$$

and by (7.16),

$$P_t\left(D_s,\Delta\right) = E_t\left[e^{-(s-\Delta-t)\delta}\frac{C_t}{C_{s-\Delta}}\left(F_{s-\Delta,s-\Delta}e^{-\Delta\delta_1}+\left(1-F_{s-\Delta,s-\Delta}\right)e^{-\Delta\delta_2}\right)D_{s-\Delta}\right] =$$

$$= E_t\left[F_{s-\Delta,s-\Delta}e^{-\Delta\delta_1}e^{-(s-\Delta-t)\delta}\frac{C_t}{C_{s-\Delta}}D_{s-\Delta}\right]+$$

$$+E_t\left[\left(1-F_{s-\Delta,s-\Delta}\right)e^{-\Delta\delta_2}e^{-(s-\Delta-t)\delta}\frac{C_t}{C_{s-\Delta}}D_{s-\Delta}\right]. \tag{7.19}$$

Next, by Assumption 3 in Section 2.2, $F_{s-\Delta,s-\Delta}$ is independent of $D_{s-\Delta}$ and $C_{s-\Delta}$ and by Assumption 2, $E_t\left[F_{s-\Delta,s-\Delta}\right] = F_{t,s-\Delta}$. Hence,

$$P_t\left(D_s,\Delta\right) = F_{t,s-\Delta}e^{-\Delta\delta_1}E_t\left[e^{-(s-\Delta-t)\delta}\frac{C_t}{C_{s-\Delta}}D_{s-\Delta}\right]+$$

$$+\left(1-F_{t,s-\Delta}\right)e^{-\Delta\delta_2}E_t\left[e^{-(s-\Delta-t)\delta}\frac{C_t}{C_{s-\Delta}}D_{s-\Delta}\right] = \tag{7.20}$$

$$= \left(F_{t,s-\Delta}e^{-\Delta\delta_1}+\left(1-F_{t,s-\Delta}\right)e^{-\Delta\delta_2}\right)E_t\left[e^{-(s-\Delta-t)\delta}\frac{C_t}{C_{s-\Delta}}D_{s-\Delta}\right]. \tag{7.21}$$

Similarly to (7.13)-(7.16) (replace s by $s-\Delta$),

$$P_{s-2\Delta}\left(D_{s-\Delta},\Delta\right) = E_{s-2\Delta}\left[e^{-(\Delta)\delta}\frac{C_{s-2\Delta}}{C_{s-\Delta}}D_{s-\Delta}\right] =$$

$$= \left(F_{s-2\Delta,s-2\Delta}e^{-\Delta\delta_1}+\left(1-F_{s-2\Delta,s-2\Delta}\right)e^{-\Delta\delta_2}\right)D_{s-2\Delta}$$

and more generally,

$$E_{s-k\Delta}\left[e^{-(\Delta)\delta}\frac{C_{s-k\Delta}}{C_{s-(k-1)\Delta}}D_{s-(k-1)\Delta}\right] = \left(F_{s-k\Delta,s-k\Delta}e^{-\Delta\delta_1}+\left(1-F_{s-k\Delta,s-k\Delta}\right)e^{-\Delta\delta_2}\right)D_{s-k\Delta},$$

$k = 1, 2, ..., \frac{s-t-\Delta}{\Delta}$. Similarly to (7.17)-(7.21) (replace s by $s-\Delta$),

$$E_t\left[e^{-(s-\Delta-t)\delta}\frac{C_t}{C_{s-\Delta}}D_{s-\Delta}\right] = \left(F_{t,s-2\Delta}e^{-\Delta\delta_1} + \left(1-F_{t,s-2\Delta}\right)e^{-\Delta\delta_2}\right)\times$$
$$\times E_t\left[e^{-(s-2\Delta-t)\delta}\frac{C_t}{C_{s-2\Delta}}D_{s-2\Delta}\right]$$

and more generally,

$$E_t\left[e^{-(s-k\Delta-t)\delta}\frac{C_t}{C_{s-k\Delta}}D_{s-k\Delta}\right] = \left(F_{t,s-(k+1)\Delta}e^{-\Delta\delta_1} + \left(1-F_{t,s-(k+1)\Delta}\right)e^{-\Delta\delta_2}\right)\times$$
$$\times E_t\left[e^{-(s-(k+1)\Delta-t)\delta}\frac{C_t}{C_{s-(k+1)\Delta}}D_{s-(k+1)\Delta}\right] \quad (7.22)$$

Finally, starting from $k=0$ and iterating (7.22) $\frac{s-t}{\Delta}$ times yields

$$P_t(D_s,\Delta) = \prod_{k=0}^{\frac{s-t}{\Delta}-1}\left(F_{t,s-(k+1)\Delta}e^{-\Delta\delta_1} + \left(1-F_{t,s-(k+1)\Delta}\right)e^{-\Delta\delta_2}\right)D_t.$$

□

Lemma 7.6. *(Price of D_s with Horizon-Invariant Beliefs) Assume $F_{t,u} = F_t$ for all $u \in (t,s)$. Then, time-t price $P_t(D_s)$ of a claim to D_s is given by*

$$P_t(D_s) = e^{-(s-t)(F_t\delta_1 + (1-F_t)\delta_2)}.$$

Proof. $P_t(D_s)$ is simply $\lim_{\Delta\to 0} P_t(D_s,\Delta)$, where

$$\lim_{\Delta\to 0} P_t(D_s,\Delta) = \lim_{\Delta\to 0} e^{\log P_t(D_s,\Delta)} =$$
$$= e^{\lim_{\Delta\to 0}\log P_t(D_s,\Delta)}.$$

By Lemma 7.5 and since $F_{t,u} = F_t$ for all $u \in (t,s)$,

$$P_t(D_s,\Delta) = \left(F_t e^{-\Delta\delta_1} + (1-F_t)e^{-\Delta\delta_2}\right)^{\frac{s-t}{\Delta}} D_t \quad (7.23)$$

and so

$$\log P_t(D_s,\Delta) = \log D_t + \frac{s-t}{\Delta}\log\left(F_t e^{-\Delta\delta_1} + (1-F_t)e^{-\Delta\delta_2}\right).$$

Note next that

$$\lim_{\Delta\to 0}\frac{s-t}{\Delta}=\infty$$

and

$$\lim_{\Delta\to 0}\log\left(F_t e^{-\Delta\delta_1}+(1-F_t)\,e^{-\Delta\delta_2}\right)=\log 1=0.$$

Hence, by L'Hopital's rule,

$$\begin{aligned}&\lim_{\Delta\to 0}\frac{s-t}{\Delta}\log\left(F_t e^{-\Delta\delta_1}+(1-F_t)\,e^{-\Delta\delta_2}\right)=\\&=\lim_{\Delta\to 0}\frac{\log\left(F_t e^{-\Delta\delta_1}+(1-F_t)\,e^{-\Delta\delta_2}\right)}{\frac{\Delta}{s-t}}=\end{aligned}$$

$$\begin{aligned}&=\lim_{\Delta\to 0}\frac{\frac{-\delta_1 F_t e^{-\Delta\delta_1}-\delta_2(1-F_t)e^{-\Delta\delta_2}}{F_t e^{-\Delta\delta_1}+(1-F_t)e^{-\Delta\delta_2}}}{\frac{1}{s-t}}=\\&=-(s-t)\left(F_t\delta_1+(1-F_t)\,\delta_2\right).\end{aligned}$$

Finally,

$$\lim_{\Delta\to 0}\log P_t\left(D_s,\Delta\right)=\log D_t-(s-t)\left(F_t\delta_1+(1-F_t)\,\delta_2\right)$$

and so

$$P_t\left(D_s\right)=e^{-(s-t)(F_t\delta_1+(1-F_t)\delta_2)}D_t.$$

□

Corollary 7.2. *(Index Price with Horizon-Invariant Beliefs) Assume $F_{t,s}=F_t$ for all $s>t$. Then, time-t price of an index with perpetual stochastic dividends as described in Section 2.1 is given by*

$$P_t=\frac{1}{F_t\delta_1+(1-F_t)\,\delta_2}D_t.$$

Proof. The result follows from the following chain of equalities:

$$\begin{aligned} P_t &= E_t\left[\int_t^\infty e^{-(s-t)\delta}\frac{C_t}{C_s}D_s ds\right] = \\ &= \int_t^\infty E_t\left[e^{-(s-t)\delta}\frac{C_t}{C_s}D_s\right]ds = \\ &= D_t\int_t^\infty e^{-(s-t)(F_t\delta_1+(1-F_t)\delta_2)}ds = \\ &= \frac{1}{F_t\delta_1+(1-F_t)\,\delta_2}D_t. \end{aligned}$$

The first line is the fundamental asset pricing equation with the SDF derived in Section 7.1, the second line is trivial, the third follows by Lemma 7.6 and the last line is straightforward integration. □

Lemma 7.7. *(Price of D_s Under General Beliefs) Under Assumption 1 in Section 2.2, time-t price $P_t(D_s)$ of a claim to D_s is given by*

$$P_t(D_s) = e^{-(s-t)\left(\delta_1\tilde{F}_{t,s}+\delta_2\left(1-\tilde{F}_{t,s}\right)\right)}D_t,$$

where $\tilde{F}_{t,s}\equiv\sum_{i=1}^{N(s)}w_iF_t(i)$, $N(s)$ denotes the number of segments in the sub-partition of (t,s), w_i is the length of $i-th$ segment in the sub-partition of (t,s) divided by $s-t$, and $F_t(i)$ is the subjective probability of being in the good regime at any point within the $i-th$ segment.

Proof. We want to derive an expression for $\lim_{\Delta\to0}P_t(D_s,\Delta)$ under Assumption 1, where $P_t(D_s,\Delta)$ is defined in Lemma 7.5. This will be accomplished by expressing $P_t(D_s,\Delta)$ as a product of terms to which the results of Lemma 7.6 apply. Start by noting that the first segment in the partition of (t,s) is given by $[t,t+w_1(s-t))$, the second - by $[t+w_1(s-t),t+(w_1+w_2)(s-t))$ and more generally, the $i-$th segment is given by

$$\left[t+\left(\sum_{l=1}^{i-1}w_l\right)(s-t),t+\left(\sum_{l=1}^{i}w_l\right)(s-t)\right).$$

When we decompose (t,s) into disjoint intervals of length Δ (call the resulting intervals $\Delta-$intervals), $i-$th segment in the partition of (t,s) will feature $w_i\frac{s-t}{\Delta}$ $\Delta-$intervals. These observations motivate the following derivations: define $\sum_{l=1}^{0}w_l\equiv0$ and note

that

$$
\begin{aligned}
P_t\left(D_s,\Delta\right) &= \prod_{k=1}^{\frac{s-t}{\Delta}}\left(F_{t,t+(k-1)\Delta}e^{-\Delta\delta_1}+\left(1-F_{t,t+(k-1)\Delta}\right)e^{-\Delta\delta_2}\right)D_t = \\
&= \prod_{i=1}^{N(s)}\left(\prod_{k=\left(\sum_{l=1}^{i-1}w_l\right)\frac{s-t}{\Delta}+1}^{\left(\sum_{l=1}^{i}w_l\right)\frac{s-t}{\Delta}}\left(F_{t,t+(k-1)\Delta}e^{-\Delta\delta_1}+\left(1-F_{t,t+(k-1)\Delta}\right)e^{-\Delta\delta_2}\right)\right)D_t \\
&= \prod_{i=1}^{N(s)}\left(\prod_{k=\left(\sum_{l=1}^{i-1}w_l\right)\frac{s-t}{\Delta}+1}^{\left(\sum_{l=1}^{i}w_l\right)\frac{s-t}{\Delta}}\left(F_t\left(i\right)e^{-\Delta\delta_1}+\left(1-F_t\left(i\right)\right)e^{-\Delta\delta_2}\right)\right)D_t \\
&= \prod_{i=1}^{N(s)}\left(F_t\left(i\right)e^{-\Delta\delta_1}+\left(1-F_t\left(i\right)\right)e^{-\Delta\delta_2}\right)^{w_i\frac{s-t}{\Delta}}D_t.
\end{aligned}
$$

The first line is reparametrized expression from Lemma 7.5, the second line is the restatement of the first one so that the product symbol inside the brackets groups terms that fall into segment i of (t,s). The third line holds because beliefs are the same within each segment of the partition, while the fourth follows from the fact that there are $w_i\frac{s-t}{\Delta}$ $\Delta-$intervals within i-th segment. Hence,

$$
\begin{aligned}
\lim_{\Delta\to 0}P_t\left(D_s,\Delta\right) &= \prod_{i=1}^{N(s)}\lim_{\Delta\to 0}\left(F_t\left(i\right)e^{-\Delta\delta_1}+\left(1-F_t\left(i\right)\right)e^{-\Delta\delta_2}\right)^{w_i\frac{s-t}{\Delta}}D_t = \\
&= \prod_{i=1}^{N(s)}e^{-w_i(s-t)(F_t(i)\delta_1+(1-F_t(i))\delta_2)}D_t = \\
&= e^{-(s-t)\sum_{i=1}^{N(s)}w_i(F_t(i)\delta_1+(1-F_t(i))\delta_2)}D_t = \\
&= e^{-(s-t)\left(\delta_1\sum_{i=1}^{N(s)}w_iF_t(i)+\delta_2\sum_{i=1}^{N(s)}w_i(1-F_t(i))\right)}D_t = \\
&= e^{-(s-t)\left(\delta_1\sum_{i=1}^{N(s)}w_iF_t(i)+\delta_2\left(1-\sum_{i=1}^{N(s)}w_iF_t(i)\right)\right)}D_t = \\
&= e^{-(s-t)\left(\delta_1\tilde{F}_{t,s}+\delta_2\left(1-\tilde{F}_{t,s}\right)\right)}D_t,
\end{aligned}
$$

where $\tilde{F}_{t,s}\equiv\sum_{i=1}^{N(s)}w_iF_t\left(i\right)$. The first line follows by Limit Laws, the second line - by Lemma 7.6, and the rest is straightforward algebra. □

7.4.2 Proposition 2.2

The proof of Proposition 2.2 is perfectly analogous to the proof of Proposition 2.1. One only needs to replace Lemma 7.4 with Lemma 7.8 below, and all other steps immediately follow once δ_i is replaced with r_i, and D_s - with 1.

Lemma 7.8. *Suppose the economy is in regime R_i at some time $\tilde{t}$ and that the regime will persist for sure until some time $\tilde{s} > \tilde{t}$. Then, time-$\tilde{t}$ value $\tilde{q}_{\tilde{t},\tilde{s}}$ of a riskless claim to one unit of consumption at time $\tilde{s}$ equals*

$$\tilde{q}_{\tilde{t},\tilde{s}} = e^{-(\tilde{s}-\tilde{t})r_i},$$

where

$$r_i = \delta + \mu_C^{(i)} - \left(\sigma_C^{(i)}\right)^2.$$

Proof. Denote by $E_{\tilde{t}}^i[.]$ time-$\tilde{t}$ conditional expectation in a single-regime-R_i economy and note the following chain of equalities:

$$\tilde{q}_{\tilde{t},\tilde{s}} = E_{\tilde{t}}^i\left[\frac{M_{\tilde{s}}}{M_{\tilde{t}}}\right] = E_{\tilde{t}}^i\left[e^{\log\left(\frac{M_{\tilde{s}}}{M_{\tilde{t}}}\right)}\right] =$$

$$= e^{-\left(r_i+\frac{1}{2}\left(\sigma_C^{(i)}\right)^2\right)(\tilde{s}-\tilde{t})+\frac{1}{2}\left(\sigma_C^{(i)}\right)^2(\tilde{s}-\tilde{t})} = e^{-(\tilde{s}-\tilde{t})r_i}.$$

The first equation holds by Section 7.1, the second is trivial, while the last line follows by Lemma 7.1. □

7.5 Expected Volatility

Lemma 7.9. *Suppose in regime R_i prices evolve according to*

$$\frac{dP_s}{P_s} = \mu_i ds + \sigma_i dB_s \quad i = 1,2$$

and assume $F_{t,s} = F_t(1)$ for all $s \in (t, t+1)$. Then,

$$E_t\left[\int_t^{t+1}\left(\frac{dP_s}{P_s}\right)^2\right] = F_t(1)\,\sigma_1^2 + (1 - F_t(1))\,\sigma_2^2.$$

Proof. Start by noting that

$$E_t\left[\int_t^{t+1}\left(\frac{dP_s}{P_s}\right)^2\right]=\sum_{k=0}^{\frac{1}{\Delta}-1}E_t\left[\int_{t+k\Delta}^{t+(k+1)\Delta}\left(\frac{dP_s}{P_s}\right)^2\right]. \tag{7.24}$$

I will initially assume for each $k\in\{0,1,2,...,\frac{1}{\Delta}-1\}$ that a regime is determined at $t+k\Delta$ and persists for sure until $t+(k+1)\Delta$. I will derive expected variance under this assumption and then take limit as $\Delta\to 0$. In the limit, a regime can switch at any point in time and persists for sure over infinitesimal time interval only. Start by defining $R_{t+k\Delta}$ as a binary variable that equals R_1 if good regime is drawn at time $t+k\Delta$ and that equals R_2 otherwise, $k=0,2,...,\frac{1}{\Delta}-1$. Note next that because

$$\int_{t+k\Delta}^{t+(k+1)\Delta}\left(\frac{dP_s}{P_s}\right)^2=\begin{cases}\sigma_1^2\Delta, & if \quad R_{t+k\Delta}=R_1\\ \sigma_2^2\Delta, & if \quad R_{t+k\Delta}=R_2\end{cases},$$

we have by Law of Iterated Expectations that

$$E_t\left[\int_{t+k\Delta}^{t+(k+1)\Delta}\left(\frac{dP_s}{P_s}\right)^2\right]=F_t(1)\sigma_1^2\Delta+(1-F_t(1))\sigma_2^2\Delta.$$

Finally, putting the above back into (7.24), we get

$$\begin{aligned}E_t\left[\int_t^{t+1}\left(\frac{dP_s}{P_s}\right)^2\right]&=\left(F_t(1)\sigma_1^2\Delta+(1-F_t(1))\sigma_2^2\Delta\right)\frac{1}{\Delta}=\\&=F_t(1)\sigma_1^2+(1-F_t(1))\sigma_2^2.\end{aligned} \tag{7.25}$$

Because the above does not depend on Δ, (7.25) holds in the limit as well as $\Delta\to 0$ and the proof is complete. □

Corollary 7.3. *In the context of Lemma 7.9, assume $F_{t,s}=F_t(k)$ for all $s\in(t+k-1,t+k)$, $k\in\mathbb{N}$. Then,*

$$E_t\left[\int_{t+k-1}^{t+k}\left(\frac{dP_s}{P_s}\right)^2\right]=F_t(k)\sigma_1^2+(1-F_t(k))\sigma_2^2.$$

Proof. The result follows by the following chain of equalities:

$$E_t\left[\int_{t+k-1}^{t+k}\left(\frac{dP_s}{P_s}\right)^2\right] = E_t\left[E_{t+k-1}\left[\int_{t+k-1}^{t+k}\left(\frac{dP_s}{P_s}\right)^2\right]\right] =$$
$$= E_t\left[F_{t+k-1}(1)\,\sigma_1^2 + (1-F_{t+k-1}(1))\,\sigma_2^2\right] =$$
$$= F_t(k)\,\sigma_1^2 + (1-F_t(k))\,\sigma_2^2.$$

The first line holds by the Law of Iterated Expectations, the second - by Lemma 7.9, while the third - by Assumption 2 in Section 2.2. □

Lemma 7.10. *Suppose $\bar{V}_t = X_t^2$, where X_t is a zero-mean Ornstein-Uhlenbeck process described by*

$$dX_t = -\tilde{\kappa}X_t dt + \tilde{\sigma}dW_t. \tag{7.26}$$

Then,

$$d\bar{V}_t = \bar{\kappa}\left(\bar{\theta} - \bar{V}_t\right)dt + \sqrt{\bar{V}_t}\bar{\sigma}dW_t$$

and

$$E_t\left[\bar{V}_s\right] = \bar{V}_t e^{-2\tilde{\kappa}(s-t)} + \bar{\theta}\left(1 - e^{-2\tilde{\kappa}(s-t)}\right),$$

where $\bar{\kappa} \equiv 2\tilde{\kappa}$, $\bar{\sigma} \equiv 2\tilde{\sigma}$ and $\bar{\theta} \equiv \frac{\tilde{\sigma}^2}{2\tilde{\kappa}}$.

Proof. By Ito's Lemma,

$$d\bar{V}_t = d\left(X_t^2\right) = 2X_t dX_t + (dX_t)^2 =$$

$$= 2X_t\left(-\tilde{\kappa}X_t dt + \tilde{\sigma}dW_t\right) + \tilde{\sigma}^2 dt =$$
$$= 2\tilde{\kappa}\left(\frac{\tilde{\sigma}^2}{2\tilde{\kappa}} - X_t^2\right)dt + X_t 2\tilde{\sigma}dW_t.$$

Set $\bar{\kappa} \equiv 2\tilde{\kappa}$, $\bar{\sigma} \equiv 2\tilde{\sigma}$ and $\bar{\theta} \equiv \frac{\tilde{\sigma}^2}{2\tilde{\kappa}}$ and note that the above can be rewritten as

$$d\bar{V}_t = \bar{\kappa}\left(\bar{\theta} - \bar{V}_t\right)dt + \sqrt{\bar{V}_t}\bar{\sigma}dW_t.$$

Consider now $\bar{V}_s$, $s > t$. Trivially,

$$E_t\left[\bar{V}_s\right] = E_t\left[X_s^2\right] = E_t\left[X_s\right]^2 + Var_t\left(X_s\right).$$

It is well known that given time t, the solution to (7.26) for time s is given by

$$X_s = X_t e^{-\tilde{\kappa}(s-t)} + \int_t^s \tilde{\sigma} e^{\tilde{\kappa}(u-s)} dW_u$$

and so

$$E_t[X_s] = X_t e^{-\tilde{\kappa}(s-t)}$$

and

$$Var_t(X_s) = \tilde{\sigma}^2 \int_t^s e^{2\tilde{\kappa}(u-s)} du = \frac{\tilde{\sigma}^2}{2\tilde{\kappa}} \left(1 - e^{2\tilde{\kappa}(t-s)}\right) = \bar{\theta}\left(1 - e^{-2\tilde{\kappa}(s-t)}\right).$$

It is then immediate that

$$E_t\left[\bar{V}_s\right] = \bar{V}_t e^{-2\bar{\kappa}(s-t)} + \bar{\theta}\left(1 - e^{-2\bar{\kappa}(s-t)}\right).$$

□

Proposition 3.1. *Suppose in regime R_i prices evolve according to*

$$\frac{dP_s}{P_s} = \mu_i ds + \sigma_s dB_s,$$

where

$$d\left(\sigma_s^2\right) = \bar{\kappa}\left(\bar{\theta}_i - \sigma_s^2\right) dt + \sqrt{\sigma_s^2}\bar{\sigma}_i dW_t,$$

$\bar{\theta}_i \equiv \frac{\bar{\sigma}_i^2}{4\bar{\kappa}}$, $i = 1,2$. *Assume further that $F_{t,s} = F_t(1)$ for all $s \in (t, t+1)$. Then,*

$$E_t\left[\int_t^{t+1} \left(\frac{dP_s}{P_s}\right)^2\right] = w\sigma_t^2 + (1-w)\left(F_t(1)\bar{\theta}_1 + (1 - F_t(1))\bar{\theta}_2\right),$$

$w = \frac{1}{\bar{\kappa}}\left(1 - e^{-\bar{\kappa}}\right)$.

Proof. Start by noting that $\left(\frac{dP_s}{P_s}\right)^2 = \sigma_s^2 ds$ and so

$$E_t\left[\int_t^{t+1} \left(\frac{dP_s}{P_s}\right)^2\right] = E_t\left[\int_t^{t+1} \sigma_s^2 ds\right] =$$

$$= \int_t^{t+1} E_t\left[\sigma_s^2\right] ds.$$

To prove the proposition, we only need to compute $E_t\left[\sigma_s^2\right]$ and integrate w.r.t s over

$(t, t+1)$. Note, however, that the computation of $E_t\left[\sigma_s^2\right]$ is not straightforward even with Lemma 7.10, since the dynamics of σ_s^2 depend on the underlying regime R_i at any point in time. To handle this, I first decompose interval (t, s) into disjoint segments of equal length Δ and make the following assumption: a regime is determined at the start-point of each segment and persists for sure until the endpoint of the segment. I compute $E_t\left[\sigma_s^2\right]$ under this assumption and then take limit as $\Delta \to 0$. In the limit, a regime can switch at any point and persists for sure over an arbitrarily small time interval only. I value $E_t\left[\sigma_s^2\right]$ for fixed Δ recursively and using the Law of Iterated Expectations (LIE).

Start by defining $R_{s-k\Delta}$ as a binary variable that equals R_1 if good regime is drawn at time $s-k\Delta$ and that equals R_2 otherwise, $k = 1, 2, ..., \frac{s-t}{\Delta}$. Note next by LIE that

$$E_t\left[\sigma_s^2\right] = E_t\left[E_{s-\Delta}\left[\sigma_s^2\right]\right], \tag{7.27}$$

where $E_{s-\Delta}[.]$ is the expectation operator conditional on all information available at time $s-\Delta$ excluding $R_{s-\Delta}$. By another application of LIE,

$$E_{s-\Delta}\left[\sigma_s^2\right] = E_{s-\Delta}\left[E_{s-\Delta}\left[\sigma_s^2 | R_{s-\Delta}\right]\right], \tag{7.28}$$

where, by Lemma 7.10,

$$E_{s-\Delta}\left[\sigma_s^2 | R_{s-\Delta}\right] = \begin{cases} \sigma_{s-\Delta}^2 e^{-2\tilde{\kappa}\Delta} + \bar{\theta}_1\left(1 - e^{-2\tilde{\kappa}\Delta}\right), & if \quad R_{s-\Delta} = R_1 \\ \sigma_{s-\Delta}^2 e^{-2\tilde{\kappa}\Delta} + \bar{\theta}_2\left(1 - e^{-2\tilde{\kappa}\Delta}\right), & if \quad R_{s-\Delta} = R_2 \end{cases} \tag{7.29}$$

and so

$$\begin{aligned} E_{s-\Delta}\left[\sigma_s^2\right] &= F_t(1)\left(\sigma_{s-\Delta}^2 e^{-2\tilde{\kappa}\Delta} + \bar{\theta}_1\left(1 - e^{-2\tilde{\kappa}\Delta}\right)\right) + \qquad (7.30) \\ &\quad + (1 - F_t(1))\left(\sigma_{s-\Delta}^2 e^{-2\tilde{\kappa}\Delta} + \bar{\theta}_2\left(1 - e^{-2\tilde{\kappa}\Delta}\right)\right) = \\ &= e^{-2\tilde{\kappa}\Delta}\sigma_{s-\Delta}^2 + \left(1 - e^{-2\tilde{\kappa}\Delta}\right)\left(F_t(1)\bar{\theta}_1 + (1 - F_t(1))\bar{\theta}_2\right). \qquad (7.31) \end{aligned}$$

Plugging this back into (7.27) gives

$$E_t\left[\sigma_s^2\right] = \left(1 - e^{-2\tilde{\kappa}\Delta}\right)\left(F_t(1)\bar{\theta}_1 + (1 - F_t(1))\bar{\theta}_2\right) + e^{-2\tilde{\kappa}\Delta}E_t\left[\sigma_{s-\Delta}^2\right],$$

and replacing s by $s-\Delta$ in (7.27) and repeating steps (7.27)-(7.31) yields

$$E_t\left[\sigma^2_{s-\Delta}\right] = \left(1-e^{-2\tilde{\kappa}\Delta}\right)\left(F_t(1)\,\bar{\theta}_1 + (1-F_t(1))\,\bar{\theta}_2\right) + e^{-2\tilde{\kappa}\Delta}E_t\left[\sigma^2_{s-2\Delta}\right]$$

and

$$E_t\left[\sigma^2_s\right] = \left(1-e^{-2\tilde{\kappa}\Delta}\right)\left(F_t(1)\,\bar{\theta}_1 + (1-F_t(1))\,\bar{\theta}_2\right)\left(1+e^{-2\tilde{\kappa}\Delta}\right) + \left(e^{-2\tilde{\kappa}\Delta}\right)^2 E_t\left[\sigma^2_{s-2\Delta}\right]. \tag{7.32}$$

Iterating 7.32 $\frac{s-t}{\Delta}-2$ more times further gives

$$E_t\left[\sigma^2_s\right] = \left(1-e^{-2\tilde{\kappa}\Delta}\right)\left(F_t(1)\,\bar{\theta}_1 + (1-F_t(1))\,\bar{\theta}_2\right)\left(1+e^{-2\tilde{\kappa}\Delta} + \left(e^{-2\tilde{\kappa}\Delta}\right)^2 + ... + \left(e^{-2\tilde{\kappa}\Delta}\right)^{\frac{s-t}{\Delta}-1}\right) +$$
$$+\left(e^{-2\tilde{\kappa}\Delta}\right)^{\frac{s-t}{\Delta}} E_t\left[\sigma^2_t\right],$$

where

$$1+e^{-2\tilde{\kappa}\Delta} + \left(e^{-2\tilde{\kappa}\Delta}\right)^2 + ... + \left(e^{-2\tilde{\kappa}\Delta}\right)^{\frac{s-t}{\Delta}-1} = \frac{1-\left(e^{-2\tilde{\kappa}\Delta}\right)^{\frac{s-t}{\Delta}}}{1-e^{-2\tilde{\kappa}\Delta}} = \frac{1-e^{-2\tilde{\kappa}(s-t)}}{1-e^{-2\tilde{\kappa}\Delta}},$$

and so

$$E_t\left[\sigma^2_s\right] = e^{-2\tilde{\kappa}(s-t)}\sigma^2_t + \left(1-e^{-2\tilde{\kappa}(s-t)}\right)\left(F_t(1)\,\bar{\theta}_1 + (1-F_t(1))\,\bar{\theta}_2\right). \tag{7.33}$$

Note that the above holds for any Δ that divides $s-t$ and so it holds for arbitrarily small Δ. Finally,

$$\int_t^{t+1} E_t\left[\sigma^2_s\right] ds = \sigma^2_t \int_t^{t+1} e^{-2\tilde{\kappa}(s-t)} ds + \left(F_t(1)\,\bar{\theta}_1 + (1-F_t(1))\,\bar{\theta}_2\right)\int_t^{t+1}\left(1-e^{-2\tilde{\kappa}(s-t)}\right) ds$$

and because

$$\int_t^{t+1} e^{-2\tilde{\kappa}(s-t)} ds = \frac{1}{2\tilde{\kappa}}\left(1-e^{-2\tilde{\kappa}}\right) = \frac{1}{\bar{\kappa}}\left(1-e^{-\bar{\kappa}}\right) \equiv w,$$

we get

$$\int_t^{t+1} E_t\left[\sigma^2_s\right] ds = w\sigma^2_t + (1-w)\left(F_t(1)\,\bar{\theta}_1 + (1-F_t(1))\,\bar{\theta}_2\right).$$

□

Corollary 7.4. *Take the setting of Proposition 3.1 and assume $F_{t,s} = F_t(2)$ for all*

$s \in (t+1, t+2)$. *Then,*

$$E_t\left[\int_{t+1}^{t+2}\left(\frac{dP_s}{P_s}\right)^2\right] = wE_t\left[\sigma_{t+1}^2\right] + (1-w)\left(F_t(2)\,\bar{\theta}_1 + (1-F_t(2))\,\bar{\theta}_2\right),$$

where

$$E_t\left[\sigma_{t+1}^2\right] = e^{-\bar{\kappa}}\sigma_t^2 + \left(1-e^{-\bar{\kappa}}\right)\left(F_t(1)\,\bar{\theta}_1 + (1-F_t(1))\,\bar{\theta}_2\right)$$

and $w = \frac{1}{\bar{\kappa}}\left(1-e^{-\bar{\kappa}}\right)$.

Proof. Start by noting that by LIE,

$$E_t\left[\int_{t+1}^{t+2}\left(\frac{dP_s}{P_s}\right)^2\right] = E_t\left[E_{t+1}\left[\int_{t+1}^{t+2}\left(\frac{dP_s}{P_s}\right)^2\right]\right],$$

where, by Proposition 3.1 (replace t by $t+1$),

$$E_{t+1}\left[\int_{t+1}^{t+2}\left(\frac{dP_s}{P_s}\right)^2\right] = w\sigma_{t+1}^2 + (1-w)\left(F_{t+1}(1)\,\bar{\theta}_1 + (1-F_{t+1}(1))\,\bar{\theta}_2\right).$$

Taking time$t$ expectation of both sides and using Assumption 2 from Section 2.2 yields

$$E_t\left[\int_{t+1}^{t+2}\left(\frac{dP_s}{P_s}\right)^2\right] = wE_t\left[\sigma_{t+1}^2\right] + (1-w)\left(F_t(2)\,\bar{\theta}_1 + (1-F_t(2))\,\bar{\theta}_2\right).$$

□

More generally, it should be obvious from above that for $k > 1$,

$$E_t\left[\int_{t+k}^{t+k+1}\left(\frac{dP_s}{P_s}\right)^2\right] = wE_t\left[\sigma_{t+k}^2\right] + (1-w)\left(F_t(k+1)\,\bar{\theta}_1 + (1-F_t(k+1))\,\bar{\theta}_2\right),$$

where $E_t\left[\sigma_{t+k}^2\right]$ can be obtained in closed form as a function of σ_t^2 and $\bar{\theta}_i$, $i = 1, 2$.